

Tensore di curvatura

(M, g) varietà riemanniana

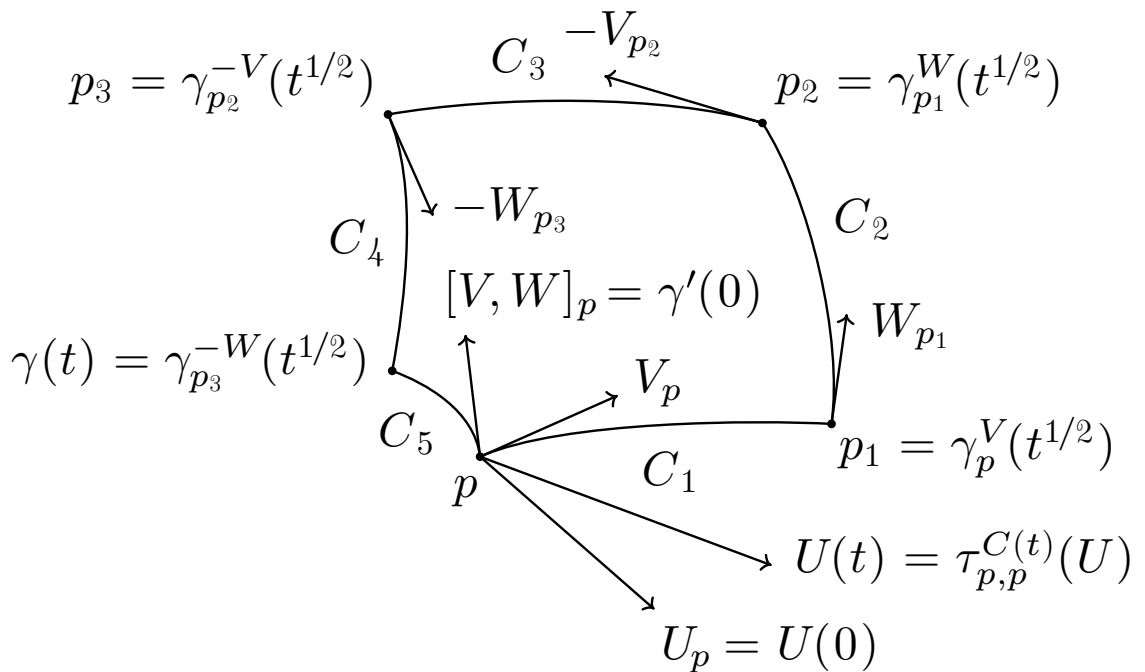
$\rightsquigarrow R : \mathcal{VM} \times \mathcal{VM} \rightarrow \text{End } \mathcal{VM}$ applicazione tale che

$$R(V, W)(U) \stackrel{\text{def}}{=} \nabla_V \nabla_W U - \nabla_W \nabla_V U - \nabla_{[V, W]} U \quad \forall V, W, U \in \mathcal{VM}$$

Note: 1) R applicazione R -bilineare antisimmetrica ben definita
 $(R(V, W) R\text{-lineare e } R(W, V) = -R(V, W) \quad \forall V, W \in \mathcal{VM})$

2) $R(V, W)(U)$ invar. per similitudini locali (poiché lo è ∇)

3) $R(V, W)(U)$ ha la seguente interpretazione geometrica
 $R(V, W)(U)_p = U'(0) = \lim_{t \rightarrow 0} (\tau_{p,p}^{C(t)=C_1 \cup \dots \cup C_5}(U_p) - U_p) / t$



Prop. (M, g) varietà riemanniana \Rightarrow

- 1) $R : \mathcal{VM} \times \mathcal{VM} \rightarrow \text{End } \mathcal{VM}$ appl. $C^\infty(M)$ -bilineare
- 2) $R(V, W) : \mathcal{VM} \rightarrow \mathcal{VM}$ appl. $C^\infty(M)$ -lineare $\forall V, W \in \mathcal{VM}$
- 3) $R(V, W)(U) + R(W, U)(V) + R(U, V)(W) = 0 \quad \forall V, W, U \in \mathcal{VM}$
identità di Bianchi \curvearrowright

Dim. 1) $R(fV, W)(U) = \nabla_{fV} \nabla_W U - \nabla_W \nabla_{fV} U - \nabla_{[fV, W]} U$
 $= f \nabla_V \nabla_W U - \nabla_W f \nabla_V U - \nabla_{f[V, W] - (Wf)V} U$
 $= f \nabla_V \nabla_W U - (Wf) \nabla_V U + f \nabla_W \nabla_V U$
 $\quad - f \nabla_{[V, W]} U + (Wf) \nabla_V U = f R(V, W)(U)$
 $R(V, fW)(U) = f R(V, W)(U)$ (segue per l'antisimmetria)

$$\begin{aligned}
2) \quad R(V, W)(fU) &= \nabla_V \nabla_W (fU) - \nabla_W \nabla_V (fU) - \nabla_{[V, W]}(fU) \\
&= \nabla_V ((Wf)U + f\nabla_W U) - \nabla_W ((Vf)U + f\nabla_V U) \\
&\quad - ([V, W]f)U - f\nabla_{[V, W]}U \\
&= (VWf)U + (Wf)\nabla_V U + (Vf)\nabla_W U + f\nabla_V \nabla_W U \\
&\quad - (WVf)U - (Vf)\nabla_W U - (Wf)\nabla_V U - f\nabla_W \nabla_V U \\
&\quad - ([V, W]f)U - f\nabla_{[V, W]}U = fR(V, W)(U) \\
3) \quad R(V, W)(U) + R(W, U)(V) + R(U, V)(W) \\
&= \nabla_V \nabla_W U - \nabla_W \nabla_V U - \nabla_{[V, W]}U \\
&\quad + \nabla_W \nabla_U V - \nabla_U \nabla_W V - \nabla_{[W, U]}V \\
&\quad + \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]}W \\
&= \nabla_V [W, U] + \nabla_W [U, V] + \nabla_U [V, W] \\
&\quad - \nabla_{[V, W]}U - \nabla_{[W, U]}V - \nabla_{[U, V]}W \\
&= [V, [W, U]] + [W, [U, V]] + [U, [V, W]] = 0
\end{aligned}$$

Note: 1) R è l'applicazione nulla se $\dim M < 2$

2) (F_1, \dots, F_m) riferimento locale

$$\begin{aligned}
V &= \sum_i v^i F_i, \quad W = \sum_j w^j F_j, \quad U = \sum_k u^k F_k \\
\Rightarrow R(V, W)(U) &= \sum_{i, j, k} v^i w^j u^k R(F_i, F_j)(F_k)
\end{aligned}$$

3) $R_{ijk} \stackrel{\text{def}}{=} R(F_i, F_j)(F_k)$ dipende solo dal riferimento

$$\begin{aligned}
\rightsquigarrow R_{ijk} &= \sum_h R_{ijk}^h F_h, \quad R_{ijkh} = g(R_{ijk}, F_h) \\
\text{t.c. } R_{ijkh} &= \sum_\ell g_{h\ell} R_{ijk}^\ell, \quad R_{ijk}^h = \sum_\ell g^{h\ell} R_{ijkl}
\end{aligned}$$

4) in coordinate locali $(F_1 = \partial/\partial x^1, \dots, F_m = \partial/\partial x^m)$

$$\begin{aligned}
R_{ijk} &= \nabla_{\frac{\partial}{\partial x^i}} \Gamma_{jk} - \nabla_{\frac{\partial}{\partial x^j}} \Gamma_{ik} = \nabla_{\frac{\partial}{\partial x^i}} \sum_h \Gamma_{jk}^h \frac{\partial}{\partial x^h} - \nabla_{\frac{\partial}{\partial x^j}} \sum_h \Gamma_{ik}^h \frac{\partial}{\partial x^h} \\
&= \sum_h \left(\frac{\partial \Gamma_{jk}^h}{\partial x^i} \frac{\partial}{\partial x^h} - \frac{\partial \Gamma_{ik}^h}{\partial x^j} \frac{\partial}{\partial x^h} + \Gamma_{jk}^h \Gamma_{ih} - \Gamma_{ik}^h \Gamma_{jh} \right) \\
&= \sum_h \left(\frac{\partial \Gamma_{jk}^h}{\partial x^i} - \frac{\partial \Gamma_{ik}^h}{\partial x^j} + \sum_\ell (\Gamma_{jk}^\ell \Gamma_{ih}^h - \Gamma_{ik}^\ell \Gamma_{jh}^h) \right) \frac{\partial}{\partial x^h} \\
\rightsquigarrow R_{ijk}^h &= \frac{\partial \Gamma_{jk}^h}{\partial x^i} - \frac{\partial \Gamma_{ik}^h}{\partial x^j} + \sum_\ell (\Gamma_{jk}^\ell \Gamma_{ih}^h - \Gamma_{ik}^\ell \Gamma_{jh}^h) \\
R_{ijkh} &= \sum_\ell \left(g_{h\ell} \left(\frac{\partial \Gamma_{jk}^\ell}{\partial x^i} - \frac{\partial \Gamma_{ik}^\ell}{\partial x^j} \right) + \Gamma_{jk}^\ell \Gamma_{ih}^h - \Gamma_{ik}^\ell \Gamma_{jh}^h \right)
\end{aligned}$$

5) $R(V, W)(U)_p$ dipende solo da $V_p, W_p, U_p \forall p \in M$

(M, g) varietà riemanniana, $p \in M$

- $\rightsquigarrow R_p : T_p M \times T_p M \rightarrow \text{End } T_p M$ appl. bilineare antisimm. t.c.
 $R_p(v, w)(u) \stackrel{\text{def}}{=} R(V, W)(U)_p \quad \forall v, w, u \in T_p M \quad \forall V, W, U \in \mathcal{V}M$
 con $V_p = v, W_p = w, U_p = u$
 \uparrow
operatore di curvatura associato alla coppia (v, w)
- $\rightsquigarrow R_p : (T_p M)^4 \rightarrow R$ forma multilineare tale che
 $R_p(v, w, u, t) \stackrel{\text{def}}{=} g_p(R_p(v, w)(u), t)$ per ogni $v, w, u, t \in T_p M$
 \uparrow
tensore di curvatura di Riemann

Prop. (M, g) varietà riemanniana, $p \in M$

- $$\left. \begin{array}{l} 1) R_p(w, v, u, t) = -R_p(v, w, u, t) \\ 2) R_p(v, w, t, u) = -R_p(v, w, u, t) \\ 3) R_p(u, t, v, w) = R_p(v, w, u, t) \end{array} \right\} \forall v, w, u, t \in T_p M$$

Dim. 1) segue dalla antisimmetria di R

- 2) basta provare $R_p(v, w, u, u) = 0 \quad \forall v, w, u \in T_p M$ (multilin.)
 quindi $g(R(V, W)(U), U) = 0 \quad \forall V, W, U \in \mathcal{V}M$ (def. di R_p)

$$\begin{aligned} g(R(V, W)(U), U) &= \\ &= g(\nabla_V \nabla_W U, U) - g(\nabla_W \nabla_V U, U) - g(\nabla_{[V, W]} U, U) \\ &= Vg(\nabla_W U, U) - g(\nabla_W U, \nabla_V U) \\ &\quad - Wg(\nabla_V U, U) + g(\nabla_V U, \nabla_W U) - \frac{1}{2}[V, W]g(U, U) \\ &= \frac{1}{2}VWg(U, U) - \frac{1}{2}WVg(U, U) - \frac{1}{2}[V, W]g(U, U) = 0 \end{aligned}$$

- 3) definizione di R_p + identità di Bianchi

$$\begin{aligned} \Rightarrow R_p(v, w, u, t) + R_p(w, u, v, t) + R_p(u, v, w, t) &= 0 \\ R_p(w, u, t, v) + R_p(u, t, w, v) + R_p(t, w, u, v) &= 0 \\ R_p(u, t, v, w) + R_p(t, v, u, w) + R_p(v, u, t, w) &= 0 \\ R_p(t, v, w, u) + R_p(v, w, t, u) + R_p(w, t, v, u) &= 0 \\ \Rightarrow 2R_p(v, w, u, t) - 2R_p(u, t, v, w) &= 0 \\ \Rightarrow R_p(u, t, v, w) = R_p(v, w, u, t) \quad \forall v, w, u, t \in T_p M \end{aligned}$$

Prop. (M, g) varietà riemanniana, $p \in M$

R_p è univoc. determinato da $R_p(v, w, v, w) \quad \forall v, w \in T_p M$

Dim. $R_p(v, w, v, w) \quad \forall v, w \in T_p M \rightsquigarrow R_p(v, w, u, w) \quad \forall v, w, u \in T_p M$

$$\begin{aligned}
 (R_p(v+u, w, v+u, w) &= R_p(v, w, v, w) + R_p(v, w, u, w) \\
 &\quad + R_p(u, w, v, w) + R_p(u, w, u, w) \\
 \rightsquigarrow R_p(v, w, u, w) &= (R_p(v+u, w, v+u, w) \\
 &\quad - R_p(v, w, v, w) - R_p(u, w, u, w))/2) \\
 R_p(v, w, u, w) \quad \forall v, w, u \in T_p M &\rightsquigarrow R_p(v, w, u, t) \quad \forall v, w, u, t \in T_p M \\
 (R_p(v, w+t, u, w+t) &= R_p(v, w, u, w) + R_p(v, w, u, t) \\
 &\quad + R_p(v, t, u, w) + R_p(v, t, u, t) \\
 \rightsquigarrow R_p(v, w, u, t) - R_p(w, u, v, t) &= a \\
 \text{con } a &= R_p(v, w+t, u, w+t) - R_p(v, w, u, w) - R_p(v, t, u, t)) \\
 \rightsquigarrow R_p(w, u, v, t) - R_p(u, v, w, t) &= b \\
 \text{con } b &= R_p(w, u+t, v, u+t) - R_p(w, u, v, u) - R_p(w, t, v, t) \\
 \rightsquigarrow \begin{cases} R_p(v, w, u, t) - R_p(w, u, v, t) = a \\ R_p(w, u, v, t) - R_p(u, v, w, t) = b \\ R_p(v, w, u, t) + R_p(w, u, v, t) + R_p(u, v, w, t) = 0 \end{cases} \\
 \rightsquigarrow R_p(v, w, u, t) &= (2a + b)/3)
 \end{aligned}$$

- Note: 1) $\{R_p\}_{p \in M} \leftrightarrow R : (\mathcal{V}M)^4 \rightarrow C^\infty(M)$ forma multilineare
t.c. $R(V, W, U, T)(p) \stackrel{\text{def}}{=} R_p(V_p, W_p, U_p, T_p) \quad \forall p \in M$, cioè
 $R(V, W, U, T) = g(R(V, W)(U), T) \quad \forall V, W, U, T \in \mathcal{V}M$
- 2) (F_1, \dots, F_m) riferimento locale intorno a $p \in M$
 $V = \sum_i v^i F_i, W = \sum_j w^j F_j, U = \sum_k u^k F_k, T = \sum_h t^h F_h$
 $\Rightarrow R(V, W, U, T) = \sum_{ijkh} R_{ijkh} v^i w^j u^k t^h$
con $R_{ijkh} = R(F_i, F_j, F_k, F_h) \quad \forall i, j, k, h = 1, \dots, m$
- 3) $R_{jihk} = R_{hkji} = R_{khij} = R_{ijkh}$
 $R_{jikh} = R_{ijhk} = R_{khji} = R_{hkij} = -R_{ijkh}$ } $\forall i, j, k, h = 1, \dots, m$
 $\Rightarrow R_{iijk} = R_{ijkk} = 0 \quad \forall i, j, k = 1, \dots, m$

(M, g) varietà riemanniana, $p \in M$

$$\rightsquigarrow S_{v,w}(u, t) \stackrel{\text{def}}{=} (R_p(v, u, w, t) + R_p(v, t, w, u))/2$$

forma bilineare simmetrica per ogni $v, w \in T_p M$

$$\rightsquigarrow S_p : T_p M \times T_p M \rightarrow R \text{ forma bilineare simmetrica}$$

tale che $S_p(v, w) \stackrel{\text{def}}{=} \text{tr}(S_{v,w})$ (teorema spettrale)

\uparrow tensore di curvatura di Ricci

Note: 1) $\{S_p\}_{p \in M} \rightsquigarrow S : \mathcal{VM} \times \mathcal{VM} \rightarrow C^\infty(M)$ forma bilineare
 t.c. $S(V, W)(p) \stackrel{\text{def}}{=} S_p(V_p, W_p) \quad \forall V, W \in \mathcal{VM} \quad \forall p \in M$

2) (F_1, \dots, F_m) riferimento locale intorno a $p \in M$

$$V = \sum_i v^i F_i, \quad W = \sum_j w^j F_j \Rightarrow S(V, W) = \sum_{ij} S_{ij} v^i w^j$$

con $S_{ij} = S(F_i, F_j) = \sum_k (R_{ikj}^k + R_{jki}^k) / 2 \quad \forall i, j = 1, \dots, m$
 $(g(\varphi_{ij}(F_k), F_h) = S_{F_i, F_j}(F_k, F_h) \Rightarrow S_{ij} = \sum_k \varphi_{ij}(F_k)^k)$

Prop. $f : (M, g^M) \rightarrow (N, g^N)$ isometria/similitudine (locale)
 con fattore di similitudine s ($= 1$ se f isometria (locale))
 $\Rightarrow R_{f(p)}^N(T_p f(v), T_p f(w), T_p f(u), T_p f(t)) = s^2 R_p^M(v, w, u, t)$
 $S_{f(p)}^N(T_p f(v), T_p f(w)) = S_p^M(v, w) \quad \forall v, w, u, t \in T_p M$

Dim. ∇ invar. per simil. (locali) $\Rightarrow R(\cdot, \cdot)$ invar. per simil. (locali)
 $\Rightarrow g_{f(p)}^N(R_{f(p)}^N(\cdot, \cdot)(\cdot), \cdot) = s^2 g_p^M(R_p^M(\cdot, \cdot)(\cdot), \cdot)$
 $S_{f(p)}^N(\cdot, \cdot) = \text{tr}(S_{(\cdot, \cdot)}^N(\cdot, \cdot)) = \text{tr}(S_{(\cdot, \cdot)}^M(\cdot, \cdot)) = S_p^M(\cdot, \cdot)$

Curvature

(M, g) varietà riemanniana, $\dim M \geq 2$

$\sigma = \langle v_1, v_2 \rangle \subset T_p M$ sezione piana (v_1 e v_2 lin. indep.)

$$\rightsquigarrow K(\sigma) \stackrel{\text{def}}{=} - \frac{R_p(v_1, v_2, v_1, v_2)}{\text{Area}(v_1, v_2)^2} = - \frac{R_p(v_1, v_2, v_1, v_2)}{g_p(v_1, v_1) g_p(v_2, v_2) - g_p(v_1, v_2)^2}$$

\nwarrow curvatura sezionale della sezione σ

Note: 1) $K(\sigma)$ è ben definita, non dipende dalla base (v_1, v_2)

$$(\sigma = \langle w_1, w_2 \rangle \Rightarrow (w_1, w_2) = M(v_1, v_2))$$

$$\Rightarrow R_p(w_1, w_2, w_1, w_2) = \det M^2 R_p(v_1, v_2, v_1, v_2)$$

$$\text{Area}(w_1, w_2) = \det M \text{Area}(v_1, v_2)$$

2) $\{v_1, v_2\}$ base ortonormale di $\sigma \Rightarrow K(\sigma) = -R(v_1, v_2, v_1, v_2)$

3) $K(\sigma) \quad \forall \sigma \subset T_p M \rightsquigarrow R_p(v, w, u, t) \quad \forall v, w, u, t \in T_p M$

4) $m = 2 \Rightarrow \sigma = T_p M \rightsquigarrow K(\sigma) = K(p)$ curvatura di Gauss

$\lambda = \langle v \rangle \subset T_p M$ direzione tangente ($v \neq 0$)

$$\rightsquigarrow K(\lambda) \stackrel{\text{def}}{=} - \frac{1}{m-1} \frac{S_p(v, v)}{\|v\|^2}$$

\nwarrow curvatura di Ricci nella direzione λ

$$p \in M \rightsquigarrow K(p) = -\frac{1}{m(m-1)} \text{tr}(S_p) \text{ (teorema spettrale)}$$

\nwarrow curvatura scalare nel punto p

Note: 1) $K(\lambda)$ è ben definita, non dipende dal vettore v

$$(\lambda = \langle w \rangle \Rightarrow w = kv \Rightarrow S_p(w, w) = k^2 S_p(v, v), \|w\| = |k| \|v\|)$$

2) $\lambda = \langle v \rangle$ con $v \in T_p M$ versore

$\{v_1, \dots, v_m\}$ base ortonorm. di $T_p M$ tale che $v_m = v$

$$\Rightarrow K(\lambda) = \frac{1}{m-1} \sum_{i=1}^{m-1} K(\sigma_{im}) \text{ con } \sigma_{im} = \langle v_i, v_m \rangle = \langle v_i, v \rangle$$

3) $\{v_1, \dots, v_m\}$ base ortonorm. di $T_p M$

$$K(p) = \frac{1}{m} \sum_{i=1}^m K(\lambda_i) \text{ con } \lambda_i = \langle v_i \rangle$$

$$= \frac{2}{m(m-1)} \sum_{i < j} K(\sigma_{ij}) \text{ con } \sigma_{ij} = \langle v_i, v_j \rangle$$

4) $m = 2 \Rightarrow K(\lambda) = K(p)$ curvatura di Gauss $\forall \lambda \subset T_p M$

$m = 3 \Rightarrow K(\lambda) \forall \lambda \subset T_p M \rightsquigarrow K(\sigma) \forall \sigma \subset T_p M$

($\{v_1, v_2, v_3\}$ base ortonorm. di $T_p M$)

$$\rightsquigarrow \begin{cases} K(\sigma_{12}) + K(\sigma_{13}) = 2K(\lambda_1) \\ K(\sigma_{12}) + K(\sigma_{23}) = 2K(\lambda_2) \\ K(\sigma_{13}) + K(\sigma_{23}) = 2K(\lambda_3) \end{cases}$$

$$\rightsquigarrow K(\sigma_{12}) = K(\lambda_1) + K(\lambda_2) - K(\lambda_3)$$

Prop. (M, g) varietà riemanniana, $\sigma \subset T_p M$ sezione piana

$$\Rightarrow K(\sigma) = K^S(p) \text{ con } S = S_\sigma = e_p(\sigma \cap B(0, \varepsilon_p)) \subset M$$

Dim. (x^1, \dots, x^m) coord. norm. su (A_p, φ_p) t.c. $\sigma = \langle \partial/\partial x^1, \partial/\partial x^2 \rangle$

$$\rightsquigarrow (F_1, \dots, F_m) \text{ riferimento ortonormale tale che } \left. \begin{aligned} &\langle F_1(q), F_2(q) \rangle = \langle \partial/\partial x^1, \partial/\partial x^2 \rangle = T_q S \quad \forall q \in S \end{aligned} \right\} \text{ (G.-S.)}$$

per $x = 0$ (cioè in p) e $i, j = 1, 2$ si ha:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0 \quad (\gamma_v(t) = tv \text{ geodetica } \forall v = \sum_k v^k \partial/\partial x^k \in T_p M)$$

$$\Rightarrow \nabla_v \gamma'_v = \sum_{i,j} v^i v^j \Gamma_{ij} = 0 \quad \forall (v^1, \dots, v^m) \in R^m$$

$$\Rightarrow \nabla_{F_i} F_j \in T_p S \quad \left(\nabla_{F_i} F_j = \sum_{k,h} a_i^k \nabla_{\frac{\partial}{\partial x^k}} \left(a_j^h \frac{\partial}{\partial x^h} \right) = \sum_{k,h} a_i^k \frac{\partial a_j^h}{\partial x^k} \frac{\partial}{\partial x^h} \right)$$

$$\begin{aligned}
&\Rightarrow g(\nabla_{F_i}^S \nabla_{F_j}^S F_1, F_2) = g(\nabla_{F_i} \nabla_{F_j}^S F_1, F_2) \\
&= F_i g(\nabla_{F_j}^S F_1, F_2) - g(\nabla_{F_j}^S F_1, \nabla_{F_i} F_2) \\
&= F_i g(\nabla_{F_j} F_1, F_2) - g(\nabla_{F_j} F_1, \nabla_{F_i} F_2) = g(\nabla_{F_i} \nabla_{F_j} F_1, F_2) \\
&\Rightarrow R_{1212}^S = g(\nabla_{F_1}^S \nabla_{F_2}^S F_1 - \nabla_{F_2}^S \nabla_{F_1}^S F_1 - \nabla_{[F_1, F_2]}^S F_1, F_2) \\
&= g(\nabla_{F_1} \nabla_{F_2} F_1 - \nabla_{F_2} \nabla_{F_1} F_1 - \nabla_{[F_1, F_2]} F_1, F_2) = R_{1212}
\end{aligned}$$

Corol. $(N, g^N) \subset (M, g^M)$ sottovarietà riemann.

$C_v^N \subset C_v^M \quad \forall v \in T_p N$ (sottovarietà geodetica in p)

$\Rightarrow K^N(\sigma) = K^M(\sigma)$ per ogni sezione piana $\sigma \subset T_p N$

Dim. $C_v^N \subset C_v^M \quad \forall v \in T_p N \Rightarrow e_{p|B(0, \varepsilon)}^N$ restrizione di $e_{p|B(0, \varepsilon)}^M$

$\Rightarrow S_\sigma^N = e_p^N(\sigma \cap B(0, \varepsilon)) = e_p^M(\sigma \cap B(0, \varepsilon)) = S_\sigma^M$

$\Rightarrow K^N(\sigma) = K^{S_\sigma^N}(p) = K^{S_\sigma^M}(p) = K^M(\sigma)$

Nota: in particolare il corollario vale per N sottovarietà totalmente geodetica (geodetica in p per ogni $p \in N$)
(per esempio $N = \text{Fix } H$ con $H \subset \text{Isom } M$)

Esempi: 1) $R^m \rightsquigarrow g_{ij} = \delta_{ij} \Rightarrow R_{ijkl} = 0$ per ogni i, j, k, ℓ
 $\Rightarrow K(\sigma) = K(\lambda) = K(p) = 0 \quad \forall \sigma, \lambda \subset T_p R^m, p \in R^m$

2) $g_{ij} = s(x)^2 \delta_{ij}$ (metrica conformemente piatta) \Rightarrow
 $R_{ijij} = s \left(\frac{\partial^2 s}{(\partial x^i)^2} + \frac{\partial^2 s}{(\partial x^j)^2} \right) - \left(\frac{\partial s}{\partial x^i} \right)^2 - \left(\frac{\partial s}{\partial x^j} \right)^2 + \sum_{k \neq i, j} \left(\frac{\partial s}{\partial x^k} \right)^2$

3) $s(x)$ dipendente solo da x^m

$$\Rightarrow \begin{cases} R_{ijij} = \left(\frac{\partial s}{\partial x^m} \right)^2 & \text{per ogni } i < j < m \\ R_{imim} = s \frac{\partial^2 s}{(\partial x^m)^2} - \left(\frac{\partial s}{\partial x^m} \right)^2 & \text{per ogni } i < m \end{cases}$$

4) $H^m \rightsquigarrow s(x) = 1/x^m$

$\Rightarrow R_{ijij} = 1/(x^m)^4$ per ogni $i < j$

$\Rightarrow K(\sigma) = K(\lambda) = K(p) = -1 \quad \forall \sigma, \lambda \subset T_p H^m, p \in H^m$

5) $s(x)$ dipendente solo da ρ^2

$$\Rightarrow \frac{\partial s}{\partial x^i} = 2x^i \frac{\partial s}{\partial \rho^2}, \quad \frac{\partial^2 s}{(\partial x^i)^2} = 2 \frac{\partial s}{\partial \rho^2} + 4(x^i)^2 \frac{\partial^2 s}{(\partial \rho^2)^2}$$

$$\Rightarrow R_{ijij} = 4s \left(\frac{\partial s}{\partial \rho^2} + ((x^i)^2 + (x^j)^2) \frac{\partial^2 s}{(\partial \rho^2)^2} \right) + 4(\rho^2 - 2(x^i)^2 - 2(x^j)^2) \left(\frac{\partial s}{\partial \rho^2} \right)^2$$

6) $S^m, D^m \rightsquigarrow s(\rho^2) = 2/(1 \pm \rho^2)$
 $\Rightarrow \partial s / \partial \rho^2 = \mp 2 / (1 \pm \rho^2)^2$, $\partial^2 s / (\partial \rho^2)^2 = 4 / (1 \pm \rho^2)^3$
 $\Rightarrow R_{ijij} = \mp 16 / (1 \pm \rho^2)^4$
 $\Rightarrow K(\sigma) = K(\lambda) = K(p) = +1 \quad \forall \sigma, \lambda \subset T_p S^m, p \in S^m$
 $K(\sigma) = K(\lambda) = K(p) = -1 \quad \forall \sigma, \lambda \subset T_p D^m, p \in D^m$

Prop. $f : (M, g^M) \rightarrow (N, g^N)$ isometria/similitudine (locale)
 con fattore di similitudine s ($= 1$ se f isometria (locale))
 $\Rightarrow K^N(T_p f(\sigma)) = K^M(\sigma) / s^2$ per ogni sezione piana $\sigma \subset T_p M$
 $K^N(T_p f(\lambda)) = K^M(\lambda) / s^2$ per ogni direzione $\lambda \subset T_p M$
 $K^N(f(p)) = K^M(p) / s^2$ per ogni punto $p \in M$

Dim. $R_{f(p)}^N(\cdot, \cdot, \cdot, \cdot) = s^2 R_p^M(\cdot, \cdot, \cdot, \cdot)$, $\text{Area}_{f(p)}^N(\cdot, \cdot) = s^2 \text{Area}_p^M(\cdot, \cdot)$
 \Rightarrow la relazione vale per la curvatura sezionale
 \Rightarrow vale per le curvatures di Ricci e scalare

Esempi: 1) $S_r^m \stackrel{\text{def}}{=} rS^m \subset R^{m+1}$ sfera di raggio $r > 0$
 con la metrica indotta da R^{m+1}
 $\rightsquigarrow ds^2 = \frac{4r^4}{(r^2 + \|x\|^2)^2} \sum_i (dx^i)^2$ in coord. stereog. su S_r^m
 $\cong \frac{4r^2}{(1 + \|x\|^2)^2} \sum_i (dx^i)^2$ in coord. stereog. su S^m
 $\Rightarrow K(\sigma) = K(\lambda) = K(p) = 1/r^2 \quad \forall \sigma, \lambda \subset T_p S_r^m, p \in S_r^m$
 2) $H_r^m \stackrel{\text{def}}{=} (\text{Int } R_+^m, ds^2 = r^2 / (x^m)^2 \sum_i (dx^i)^2)$ con $r > 0$
 $\Rightarrow K(\sigma) = K(\lambda) = K(p) = -1/r^2 \quad \forall \sigma, \lambda \subset T_p H_r^m, p \in H_r^m$
 3) $D_r^m \stackrel{\text{def}}{=} (\text{Int } rB^m, ds^2 = \frac{4r^4}{(r^2 - \|x\|^2)^2} \sum_i (dx^i)^2)$
 $D_r^m \cong (\text{Int } B^m, ds^2 = \frac{4r^2}{(1 - \|x\|^2)^2} \sum_i (dx^i)^2)$
 $\Rightarrow K(\sigma) = K(\lambda) = K(p) = -1/r^2 \quad \forall \sigma, \lambda \subset T_p D_r^m, p \in D_r^m$

Sottovarietà riemanniane

$(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana, $\dim N < \dim M$

(F_1, \dots, F_m) riferimento locale adattato lungo N

$$\stackrel{\text{def}}{\iff} \left. \begin{aligned} \langle F_1(p), \dots, F_n(p) \rangle &= T_p N \\ \langle F_{n+1}(p), \dots, F_m(p) \rangle &= T_p N^\perp \end{aligned} \right\} \forall p \in A \subset N$$

(x^1, \dots, x^m) coordinate locali adattate su $A \subset M$

$$\stackrel{\text{def}}{\iff} (\partial/\partial x^1, \dots, \partial/\partial x^m)|_{A \cap N} \text{ riferimento locale adattato lungo } N$$

Nota: (x^1, \dots, x^m) coordinate locali adattate in senso diff.

$$\rightsquigarrow (\partial/\partial x^1, \dots, \partial/\partial x^m)|_N \text{ riferimento locale lungo } N$$

$$\rightsquigarrow (F_1, \dots, F_m) \text{ riferimento locale adattato lungo } N \text{ (G.-S.)}$$

$$\rightsquigarrow (x^1, \dots, x^m) \text{ coord. locali adattate (mediante } \sqcup_p e_p|_{T_p N^\perp})$$

U campo di vettori normali lungo N ($U_p \in T_p N^\perp \subset T_p M \forall p \in N$)

$\rightsquigarrow L_U : \mathcal{V}N \rightarrow \mathcal{V}N$ applicazione $C^\infty(N)$ -lineare tale che

$$L_U(V) \stackrel{\text{def}}{=} -\pi(\nabla_V U) \text{ con } \pi = \sqcup_{p \in N} \pi_p \text{ proiezione ortogonale}$$

Note: 1) $L_U(V)$ $C^\infty(N)$ -lineare rispetto a U

$$\begin{aligned} (L_{fU}(V) &= -\pi \nabla_V (fU) = -\pi((Vf)U + f \nabla_V U) \\ &= -\pi(f \nabla_V U) = f L_U(V) \forall f \in C^\infty(N)) \end{aligned}$$

2) (F_1, \dots, F_m) riferimento locale adattato

$$U = \sum_{i=n+1}^m u^i F_i, V = \sum_{j=1}^n v^j F_j$$

$$\Rightarrow L_U(V) = \sum_{i,j} u^i v^j L_{F_i}(F_j) \text{ dove}$$

$L_{F_i} F_j$ dipende solo dal riferimento

3) $L_U(V)_p$ dipende solo da V_p e $U_p \forall p \in N$

$(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana, $p \in N$

$\rightsquigarrow L_p : T_p N^\perp \rightarrow \text{End } T_p N$ applicazione t.c. $L_p(u) = L_u$ definita

$$L_u(v) \stackrel{\text{def}}{=} L_U(V)_p \forall U, V \text{ come sopra con } U_p = u \text{ e } V_p = v$$

\uparrow operatore forma associato al vettore normale u

$\rightsquigarrow L_p : T_p N^\perp \rightarrow \text{Bil } T_p N$ applicazione t.c. $L_u(v, w) \stackrel{\text{def}}{=} g_p(L_u(v), w)$

Nota: entrambe le applicazioni sono ben definite e lineari

(non dipendono dalla scelta delle estensioni U e V)

Prop. $(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana, $u \in T_p N^\perp$
 $\Rightarrow L_u(\cdot)$ operatore simm. e $L_u(\cdot, \cdot)$ forma bilineare simm.

Dim. $V \in \mathcal{V}N \Rightarrow g(U, V) = 0 \Rightarrow g(\nabla_W U, V) + g(U, \nabla_W V) = 0$
 $W \in \mathcal{V}N \Rightarrow g(U, W) = 0 \Rightarrow g(\nabla_V U, W) + g(U, \nabla_V W) = 0$
 $\Rightarrow g(L_U V, W) - g(L_U W, V) = g(U, [V, W]) = 0$

Note: 1) $C \subset M$ curva diff. regolare orientata \rightsquigarrow riferim. di Frenet
 (riferim. (F_1, \dots, F_m) lungo C def. $F_1 = T$ e $F_k = \nabla_T F_{k-1}$
 \rightsquigarrow riferim. ortonormale lungo C ottenuto con G.-S.)
 2) $N \subset M$ orientate e $n = m - 1 \rightsquigarrow L_p \stackrel{\text{def}}{=} L_u$ con
 $u =$ unico versore normale positivo rispetto alle orient.
 \rightsquigarrow direzioni principali e forma normale (teor. spettrale)
 curv. principali, curv. media, curv. di Gauss-Kronecker
 (oper. forma e II forma fond. per superf. orient. in R^3)

Prop. $(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana
 $\Rightarrow L_u(v, w) = g_p(\nabla_v^M W - \nabla_v^N W, u)$
 $\forall u \in T_p N^\perp, v \in T_p N, W \in \mathcal{V}N$ con $W_p = w$

Dim. U campo di vettori normali lungo N tale che $U_p = u$
 $\rightsquigarrow g_p(\nabla_v^M W - \nabla_v^N W, u)$
 $= g_p(\nabla_v^M W, u) = v g(W, U) - g_p(w, \nabla_v^M U)$
 $= -g_p(w, \nabla_v^M U) = g_p(w, L_u(v)) = L_u(v, w)$

$(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana, $p \in N$
 $\rightsquigarrow L_p : T_p N \times T_p N \rightarrow T_p N^\perp$ applicazione tale che
 $L_p(v, w) \stackrel{\text{def}}{=} \nabla_v^M W - \nabla_v^N W \quad \forall W \in \mathcal{V}N$ con $W_p = w$

Note: 1) L_p è ben definita (non dipende dalla scelta di W)
 bilineare e simmetrica (proposizione precedente)

2) (F_1, \dots, F_m) riferimento locale adattato

$$v = \sum_{i=1}^n v^i F_i, w = \sum_{j=1}^n w^j F_j$$

$$\Rightarrow L_p(v, w) = \sum_{i,j} v^i w^j L_p(F_i, F_j)$$

3) $L_{ij} \stackrel{\text{def}}{=} L(F_i, F_j)$ campo di vettori normali lungo N
 che dipende solo dal riferimento

$$\begin{aligned} \rightsquigarrow L_{ij} &= \sum_{k=n+1}^m \ell_{ij}^k F_k, \quad \ell_{ijk} = g(L_{ij}, F_k) \\ \text{t.c. } \ell_{ijk} &= \sum_{h=n+1}^m g_{kh} \ell_{ij}^h, \quad \ell_{ij}^k = \sum_{h=n+1}^m g^{kh} \ell_{ijh} \\ 4) (\ell_{ijk})_{i,j} &\text{ matrice di } L_{F_k}(\cdot, \cdot) \text{ rispetto al rif. } (F_1, \dots, F_n) \end{aligned}$$

Corol. $(N, g^N) \subset (M, g^M)$ sottovarietà geodetica in $p \in N$

$$\begin{aligned} \Leftrightarrow L_u(v) &= 0 \quad \forall u \in T_p N^\perp, v \in T_p N \\ \Leftrightarrow L_u(v, w) &= 0 \quad \forall u \in T_p N^\perp, v, w \in T_p N \\ \Leftrightarrow L_p(v, w) &= 0 \quad \forall v, w \in T_p N \\ \Leftrightarrow \nabla_v^N W &= \nabla_v^M W \quad \forall v \in T_p N, W \in \mathcal{V}N \end{aligned}$$

Dim. N geodetica in $p \in N \Leftrightarrow \gamma_v^N = \gamma_v^M \quad \forall v \in T_p N$

$$\begin{aligned} \Leftrightarrow \nabla_v^N V &= \nabla_v^M V = 0 \text{ con } V = d\gamma_v^N(t)/dt \text{ campo vett. su } C_v^N \\ \Leftrightarrow L_u(v, v) &= 0 \quad \forall u \in T_p N^\perp, v \in T_p N \text{ (proposizione sopra)} \\ \Leftrightarrow L_u(v, w) &= 0 \quad \forall u \in T_p N^\perp, v, w \in T_p N \text{ (simmetria)} \\ &\text{(che equivale a } L_u(v) = 0 \quad \forall u \in T_p N^\perp, v \in T_p N) \\ \Leftrightarrow L_p(v, w) &= 0 \quad \forall v, w \in T_p N \text{ (arbitrarietà di } u) \\ \Leftrightarrow \nabla_v^N W &= \nabla_v^M W \quad \forall v \in T_p N, W \in \mathcal{V}N \text{ (definizione)} \end{aligned}$$

Prop. $(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana

tale che $\dim M > \dim N \geq 2$, $\sigma = \langle v_1, v_2 \rangle \subset T_p N \Rightarrow$

$$K^N(\sigma) = K^M(\sigma) + \frac{g_p(L_p(v_1, v_1), L_p(v_2, v_2)) - \|L_p(v_1, v_2)\|_p^2}{g_p(v_1, v_1) g_p(v_2, v_2) - g_p(v_1, v_2)^2}$$

Dim. $V_1, V_2 \in \mathcal{V}N$ tali che $V_1(p) = v_1$ e $V_2(p) = v_2 \rightsquigarrow$

$$\begin{aligned} &(K^N(\sigma) - K^M(\sigma))(g_p(v_1, v_1) g_p(v_2, v_2) - g_p(v_1, v_2)^2) \\ &= R^M(v_1, v_2, v_1, v_2) - R^N(v_1, v_2, v_1, v_2) \\ &= g_p(\nabla_{V_1}^M \nabla_{V_2}^M V_1 - \nabla_{V_2}^M \nabla_{V_1}^M V_1 - \nabla_{[V_1, V_2]}^M V_1, V_2) \\ &\quad - g_p(\nabla_{V_1}^N \nabla_{V_2}^N V_1 - \nabla_{V_2}^N \nabla_{V_1}^N V_1 - \nabla_{[V_1, V_2]}^N V_1, V_2) \\ &= g_p(\nabla_{V_1}^M \nabla_{V_2}^M V_1 - \nabla_{V_2}^M \nabla_{V_1}^M V_1 - \nabla_{V_1}^N \nabla_{V_2}^N V_1 + \nabla_{V_2}^N \nabla_{V_1}^N V_1, V_2) \\ &= g_p(\nabla_{V_1}^M \nabla_{V_2}^M V_1 - \nabla_{V_2}^M \nabla_{V_1}^M V_1 - \nabla_{V_1}^M \nabla_{V_2}^N V_1 + \nabla_{V_2}^M \nabla_{V_1}^N V_1, V_2) \\ &= g_p(\nabla_{V_1}^M (\nabla_{V_2}^M V_1 - \nabla_{V_2}^N V_1), V_2) - g_p(\nabla_{V_2}^M (\nabla_{V_1}^M V_1 - \nabla_{V_1}^N V_1), V_2) \\ &= v_1 g(L(V_1, V_2), V_2) - g_p(L_p(v_1, v_2), \nabla_{v_1}^M V_2) \\ &\quad - v_2 g(L(V_1, V_1), V_2) + g_p(L_p(v_1, v_1), \nabla_{v_2}^M V_2) \\ &= g_p(L_p(v_1, v_1), \nabla_{v_2}^M V_2) - g_p(L_p(v_1, v_2), \nabla_{v_1}^M V_2) \end{aligned}$$

$$\begin{aligned}
 &= g_p(L_p(v_1, v_1), \nabla_{v_2}^M V_2 - \nabla_{v_2}^N V_2) - g_p(L_p(v_1, v_2), \nabla_{v_1}^M V_2 - \nabla_{v_1}^N V_2) \\
 &= g_p(L_p(v_1, v_1), L_p(v_2, v_2)) - g_p(L_p(v_1, v_2), L_p(v_1, v_2))
 \end{aligned}$$

Note: 1) (F_1, \dots, F_m) riferim. adattato t.c. $F_1(p) = v_1$ e $F_2(p) = v_2$

$$\begin{aligned}
 \rightsquigarrow K^N(\sigma) &= K^M(\sigma) + \frac{g_p(L_{11}, L_{22}) - \|L_{12}\|^2}{g_{11}g_{22} - g_{12}^2} \\
 &= K^M(\sigma) + \frac{\sum_{i,j=n+1}^m g_{ij}(\ell_{11}^i \ell_{22}^j - \ell_{12}^i \ell_{12}^j)}{g_{11}g_{22} - g_{12}^2}
 \end{aligned}$$

2) $N \subset M$ orientate e $n = m - 1 \rightsquigarrow \ell_{ij} \stackrel{\text{def}}{=} \ell_{ij}^m$ con $F_m =$ unico versore normale positivo rispetto alle orient.

$$\rightsquigarrow K^N(\sigma) = K^M(\sigma) + \frac{\ell_{11}\ell_{22} - \ell_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

3) in particolare, N ipersuperficie in R^m

$$\rightsquigarrow K(\sigma) = \frac{\ell_{11}\ell_{22} - \ell_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

Forme di connessione e curvatura

(M, g) varietà riemanniana, (F_1, \dots, F_m) riferimento locale $(\varphi^1, \dots, \varphi^m)$ forme duali (in senso lineare, cioè $\varphi^i(F_j) = \delta_j^i$)

$\rightsquigarrow \psi_i^j, \psi_{ij}$ 1-forme diff. tali che $\psi_i^j(V) = \varphi^j(\nabla_V F_i)$

\uparrow forme di connessione $\psi_{ij}(V) = g(\nabla_V F_i, F_j)$
 $\forall V$ campo vett. $\forall i, j = 1, \dots, m$

ω_i^j, ω_{ij} 2-forme diff. tali che $\omega_i^j(V, W) = \varphi^j(R(V, W)(F_i))$

\uparrow forme di curvatura $\omega_{ij}(V, W) = g(R(V, W)(F_i), F_j)$
 $\forall V, W$ campi vett. $\forall i, j = 1, \dots, m$

Note: 1) $\left. \begin{aligned} \psi_i^j &= \sum_k g^{jk} \psi_{ik}, \omega_i^j = \sum_k g^{jk} \omega_{ik} \\ \psi_{ij} &= \sum_k g_{jk} \psi_i^k, \omega_{ij} = \sum_k g_{jk} \omega_i^k \end{aligned} \right\} \forall i, j$

2) $\left. \begin{aligned} \psi_i^j(F_k) &= \Gamma_{ki}^j, \omega_i^j(F_k, F_h) = R_{khi}^j \\ \psi_{ij}(F_k) &= \Gamma_{kij}, \omega_{ij}(F_k, F_h) = R_{ijkh} \end{aligned} \right\} \forall i, j, k, h$

3) $\left. \begin{aligned} \psi_i^j &= \sum_k \Gamma_{ki}^j \varphi^k, \omega_i^j = \sum_{k < h} R_{khi}^j \varphi^k \wedge \varphi^h \\ \psi_{ij} &= \sum_k \Gamma_{kij} \varphi^k, \omega_{ij} = \sum_{k < h} R_{ijkh} \varphi^k \wedge \varphi^h \end{aligned} \right\} \forall i, j$

4) (F_1, \dots, F_m) riferimento locale ortonormale

$$\Rightarrow \left. \begin{aligned} \psi_j^i &= \psi_{ji} = -\psi_{ij} = -\psi_i^j \\ \omega_j^i &= \omega_{ji} = -\omega_{ij} = -\omega_i^j \end{aligned} \right\} \forall i, j$$

Prop. (M, g) varietà riemanniana, (F_1, \dots, F_m) riferimento locale
 \Rightarrow valgono le seguenti equazioni strutturali

$$\left. \begin{aligned} 1) \quad dg_{ij} &= \psi_{ij} + \psi_{ji} \\ 2) \quad d\varphi^i &= \sum_k \varphi^k \wedge \psi_k^i \\ 3) \quad d\psi_i^j &= \omega_i^j + \sum_k \psi_i^k \wedge \psi_k^j \end{aligned} \right\} \forall i, j = 1, \dots, m$$

Dim. 1) $dg_{ij}(V) = Vg(F_i, F_j) = g(\nabla_V F_i, F_j) + g(F_i, \nabla_V F_j)$
 $= \psi_{ij}(V) + \psi_{ji}(V) \quad \forall V$ campo di vettori

$$\begin{aligned} 2) \quad \sum_k \varphi^k \wedge \psi_k^i(V, W) &= \sum_k (\varphi^k(V)\psi_k^i(W) - \varphi^k(W)\psi_k^i(V)) \\ &= \sum_k (v^k \varphi^i(\nabla_W F_k) - w^k \varphi^i(\nabla_V F_k)) \\ &= \varphi^i(\sum_k (v^k \nabla_W F_k - w^k \nabla_V F_k)) \\ &= \varphi^i(\sum_k (\nabla_W (v^k F_k) - (Wv^k)F_k - \nabla_V (w^k F_k) + (Vw^k)F_k)) \\ &= \varphi^i(\nabla_W V - \nabla_V W) - Wv^i + Vw^i \\ &= V\varphi^i(W) - W\varphi^i(V) - \varphi^i([V, W]) \\ &= d\varphi^i(V, W) \quad \forall V, W \text{ campi di vettori} \end{aligned}$$

$$\begin{aligned} 3) \quad \omega_i^j(V, W) &= \varphi^j(R(V, W)(F_i)) \\ &= \varphi^j(\nabla_V \nabla_W F_i - \nabla_W \nabla_V F_i - \nabla_{[V, W]} F_i) \\ &= \varphi^j(\sum_k (\nabla_V (\psi_i^k(W)F_k) - \nabla_W (\psi_i^k(V)F_k))) - \psi_i^j([V, W]) \\ &= \varphi^j(\sum_k ((V\psi_i^k(W))F_k + \psi_i^k(W)\nabla_V F_k \\ &\quad - (W\psi_i^k(V))F_k - \psi_i^k(V)\nabla_W F_k)) - \psi_i^j([V, W]) \\ &= V\psi_i^j(W) + \sum_k \psi_i^k(W)\psi_k^j(V) \\ &\quad - W\psi_i^j(V) - \sum_k \psi_i^k(V)\psi_k^j(W) - \psi_i^j([V, W]) \\ &= d\psi_i^j(V, W) - \sum_k \psi_i^k \wedge \psi_k^j(V, W) \\ &= (d\psi_i^j - \sum_k \psi_i^k \wedge \psi_k^j)(V, W) \quad \forall V, W \text{ campi di vettori} \end{aligned}$$

Varietà a curvatura costante

$M = (M, g)$ varietà riemanniana, $\dim M \geq 2$

a curvatura (sezionale) costante $K \stackrel{\text{def}}{\iff} K(\sigma) = K$ per ogni $\sigma \subset TM$

Prop. $M = (M, g)$ varietà riemanniana connessa, $\dim M \geq 3$
 $K(\sigma) = K(p) \forall \sigma \subset T_p M, p \in M \Rightarrow K(p) = K$ costante
 (curvatura sezionale isotropa \Rightarrow costante)

Dim. (F_1, \dots, F_m) riferimento locale ortonormale

$$\rightsquigarrow R_{ijkl}(p) = -K(p)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \quad \forall i, j, k, l = 1, \dots, m$$

(stesse simmetrie e stesse curvatures sezionali)

$$\begin{aligned} \Rightarrow \omega_{ij} &= -K \sum_{k < h} (\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}) \varphi^k \wedge \varphi^h = -K \varphi^i \wedge \varphi^j \quad \forall i < j \\ \Rightarrow d\psi_i^j &= \omega_i^j + \sum_k \psi_i^k \wedge \psi_k^j = -K \varphi^i \wedge \varphi^j + \sum_k \psi_i^k \wedge \psi_k^j \quad \forall i < j \\ \Rightarrow -dK \wedge \varphi^i \wedge \varphi^j - K d\varphi^i \wedge \varphi^j + K \varphi^i \wedge d\varphi^j + \sum_k (d\psi_i^k \wedge \psi_k^j - \psi_i^k \wedge d\psi_k^j) \\ &= -dK \wedge \varphi^i \wedge \varphi^j - K \sum_k \varphi^k \wedge \psi_k^i \wedge \varphi^j + K \sum_k \varphi^i \wedge \varphi^k \wedge \psi_k^j \\ &\quad + \sum_k (\omega_i^k \wedge \psi_k^j - \psi_i^k \wedge \omega_k^j) + \sum_{k, h} (\psi_i^h \wedge \psi_h^k \wedge \psi_k^j - \psi_i^k \wedge \psi_k^h \wedge \psi_h^j) \\ &= -dK \wedge \varphi^i \wedge \varphi^j = 0 \quad \forall i < j \end{aligned}$$

$$\Rightarrow dK \wedge \varphi^i \wedge \varphi^j = \sum_h (dK)_h \varphi^h \wedge \varphi^i \wedge \varphi^j = 0 \quad \forall i < j$$

$$\Rightarrow (dK)_h = 0 \quad \forall h = 1, \dots, m$$

$$\Rightarrow dK = 0 \Rightarrow K \text{ loc. costante} \Rightarrow K \text{ costante (} M \text{ connessa)}$$

Spazi modello: $M_K^m = \begin{cases} R^m \rightsquigarrow K(\sigma) = K = 0 \\ S_{1/\sqrt{K}}^m \rightsquigarrow K(\sigma) = K > 0 \\ H_{1/\sqrt{-K}}^m \cong D_{1/\sqrt{-K}}^m \rightsquigarrow K(\sigma) = K < 0 \end{cases}$

Teorema di Cartan (locale)

$M = (M, g)$ varietà riemanniana, $\dim M = m \geq 2$

M a curvatura costante $K \Leftrightarrow M$ localmente isometrica a M_K^m

$(A_p = B(p, \varepsilon_p) \subset M$ isometrico a $B(p_0, \varepsilon_p) \subset M_K^m \forall p \in M)$

Dim. (x^1, \dots, x^m) coordinate normali su $A_p \subset M$

$$\rightsquigarrow V = \rho \frac{\partial}{\partial \rho} = \sum_i x^i \frac{\partial}{\partial x^i} \text{ campo di vettori radiale}$$

$$\text{tale che } \nabla_V V = V \text{ e } \left[V, \frac{\partial}{\partial x^j} \right] = -\frac{\partial}{\partial x^j} \quad \forall j = 1, \dots, m$$

$\rightsquigarrow (F_1, \dots, F_m)$ riferimento locale ortonormale

tale che $\nabla_V F_i = 0$ e $F_i(p) = \partial/\partial x^i$

$$\rightsquigarrow (\varphi^1, \dots, \varphi^m) \text{ forme duali } \rightsquigarrow \psi_i^j = \psi_{ij}, \omega_i^j = \omega_{ij}$$

$$\begin{aligned} \Rightarrow \left. \begin{aligned} \varphi^k(V) &= x^k \\ \psi_j^k(V) &= \varphi^k(\nabla_V F_j) = 0 \end{aligned} \right\} \forall j, k = 1, \dots, m \\ (V\varphi^k(V) = Vg(V, F_k) = g(\nabla_V V, F_k) + g(V, \nabla_V F_k) = \varphi^k(V) \\ \Rightarrow \varphi^k(V) \text{ omog. di grado 1 t.c. } \partial\varphi^k(V)/\partial x^i = \delta_i^k \text{ in } 0) \end{aligned}$$

$$a_i^k = \varphi^k(\partial/\partial x^i) \text{ con } i, j = 1, \dots, m$$

$$b_{ij}^k = \psi_j^k(\partial/\partial x^i) \text{ con } i, j, k = 1, \dots, m$$

$$\begin{aligned} \Rightarrow Va_i^k &= d\varphi^k\left(V, \frac{\partial}{\partial x^i}\right) + \frac{\partial}{\partial x^i}\varphi^k(V) + \varphi^k\left(\left[V, \frac{\partial}{\partial x^i}\right]\right) \\ &= \sum_h (\varphi^h \wedge \psi_h^k)\left(V, \frac{\partial}{\partial x^i}\right) + \frac{\partial x^k}{\partial x^i} - \varphi^k\left(\frac{\partial}{\partial x^i}\right) \\ &= \sum_h \left(\varphi^h(V)\psi_h^k\left(\frac{\partial}{\partial x^i}\right) - \varphi^h\left(\frac{\partial}{\partial x^i}\right)\psi_h^k(V)\right) + \delta_i^k - a_i^k \\ &= \delta_i^k - a_i^k + \sum_h x^h b_{ih}^k \end{aligned}$$

$$\begin{aligned} Vb_{ij}^k &= d\psi_j^k\left(V, \frac{\partial}{\partial x^i}\right) + \frac{\partial}{\partial x^i}\psi_j^k(V) + \psi_j^k\left(\left[V, \frac{\partial}{\partial x^i}\right]\right) \\ &= \omega_j^k\left(V, \frac{\partial}{\partial x^i}\right) + \sum_h (\psi_j^h \wedge \psi_h^k)\left(V, \frac{\partial}{\partial x^i}\right) \\ &\quad + \frac{\partial}{\partial x^i}\psi_j^k(V) - \psi_j^k\left(\frac{\partial}{\partial x^i}\right) \\ &= \omega_{jk}\left(\sum_h x^h F_h, \sum_\ell a_i^\ell F_\ell\right) - b_{ij}^k \\ &= -b_{ij}^k + \sum_{h,\ell} R_{jkh\ell} x^h a_i^\ell \\ &= -b_{ij}^k - \sum_{h,\ell} K(\delta_{jh}\delta_{k\ell} - \delta_{j\ell}\delta_{kh}) x^h a_i^\ell \\ &= -b_{ij}^k - K(x^j a_i^k - x^k a_i^j) \end{aligned}$$

$v \in T_p M$ versore $\rightsquigarrow x = \rho v$ raggio uscente da p con $\rho < \varepsilon_p$

$$\rightsquigarrow f_i^k(\rho) = \rho a_i^k(\rho v), \quad f_{ij}^k(\rho) = \rho b_{ij}^k(\rho v)$$

\rightsquigarrow problema di Cauchy dipendente solo da K

$$\left\{ \begin{aligned} \frac{df_i^k}{d\rho} &= a_i^k(\rho v) + Va_i^k(\rho v) = \delta_i^k + \sum_h v^h f_{ih}^k \\ \frac{df_{ij}^k}{d\rho} &= b_{ij}^k(\rho v) + Vb_{ij}^k(\rho v) = -K(v^j f_i^k - v^k f_i^j) \\ f_i^k(0) &= f_{ij}^k(0) = 0 \end{aligned} \right.$$

$\Rightarrow f_i^k, f_{ij}^k, a_i^k, b_{ij}^k, ds_g^2 = \sum_k (\varphi^k)^2 = \sum_{i,j,k} a_i^k a_j^k dx^i dx^j$
 univocamente determinate da K in A_p

$\rightsquigarrow f : B(p_0, \varepsilon_p) \rightarrow A_p$ con $B(p_0, \varepsilon_p) \subset M_K^m$ isometria
 (con $T_{p_0} f : T_{p_0} M_K^m \rightarrow T_p M$ isometria arbitraria)
 definita dall'identità in coordinate normali

($\varepsilon_p \leq \varepsilon_{p_0}$ con $\varepsilon_{p_0} = \infty$ se $K \leq 0$ e $\varepsilon_{p_0} = \pi/\sqrt{K}$ se $K > 0$)

Note: 1) M_K^m varietà riem. omogenea isotropa per ogni K e $m \geq 2$
 ($\text{Isom}_p M_K^m \cong O(m)$ agisce trans. sui riferim. ortonorm.)

2) M a curv. costante \Rightarrow localmente omogenea e isotropa
 ($\text{Isom}_p B(p, \varepsilon) \cong O(m)$ agisce trans. sui rifer. ortonorm.)

Teorema di Cartan (globale)

$M = (M, g)$ varietà riemanniana, $\dim M = m \geq 2$

M sempl. connessa completa a curv. costante $K \Leftrightarrow M \cong M_K^m$

Dim. M completa a curvatura costante K

$\rightsquigarrow f_{p_0} : A_{p_0} \rightarrow M$ isometria locale con

$A_{p_0} = M_K^m$ se $K \leq 0$ e $A_{p_0} = M_K^m - \{-p_0\}$ se $K > 0$

$\rightsquigarrow f : M_K^m \rightarrow M$ isometria locale

($f = f_{p_1} \cup f_{p_2}$ con p_1 e p_2 non antipodali se $K > 0$)

M connessa (e M_K^m completa) $\Rightarrow f$ rivestimento

M semplicemente connessa $\Rightarrow f$ isometria

Corol. $M = (M, g)$ varietà riemanniana, $\dim M = m \geq 2$

M connessa completa a curvatura costante K

$\Leftrightarrow M \cong M_K^m / G$ con $G \subset \text{Isom } M_K^m$ prop. discontinuo

Dim. \Rightarrow) $f : \widetilde{M} \rightarrow M \cong \widetilde{M} / G_f$ rivestimento universale

$\rightsquigarrow \widetilde{M} = (\widetilde{M}, \widetilde{g})$ semplicemente connessa

tale che f isometria locale e $G_f \subset \text{Isom } \widetilde{M}$

$\Rightarrow \widetilde{M}$ completa a curvatura costante $K \Rightarrow \widetilde{M} \cong M_K^m$

\Leftarrow) $M \cong M_K^m / G$ con $G \subset \text{Isom } M_K^m$

$\Rightarrow \pi : M_K^m \rightarrow M$ rivestimento localmente isometrico

$\Rightarrow M$ connessa completa a curvatura costante K