BRAIDING NON-ORIENTABLE SURFACES IN $S^4$

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Abstract

Closed braided surfaces in $S^4$ are the two-dimensional analogous of closed braids in $S^3$. They are useful in studying smooth closed orientable surfaces in $S^4$, since any such a surface is isotopic to a braided one. We show that the non-orientable version of this result does not hold, that is smooth closed non-orientable surfaces cannot be braided. In fact, any reasonable definition of non-orientable braided surfaces leads to very strong restrictions in terms of self-intersection and Euler characteristic.

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Introduction

The concept of braided surface in $B^4 \cong B^2 \times B^2$ has been introduced since the early eighties by Rudolph (cf. [18], [19] and [20]) as a two dimensional analogous of the classical Artin’s braids. Namely, he called braided a surface in $B^2 \times B^2$ which projects onto the first factor by a branched covering.

Successively, in the nineties, Viro and Kamada (cf. [8], [9] and [11]) considered closed braided surfaces in $S^4$, that is surfaces contained in a normal neighborhood of $S^2 \subset S^4$, projecting onto $S^2$ by a branched covering. We can think of a closed braided surface as closure of a Rudolph’s braided surface with trivial boundary, just in the same way we think of a closed braid in $S^3$ as closure of an Artin’s braid.

By Kamada’s results, closed braided surfaces can be used to study orientable smooth surfaces in $S^3$. In fact, he provided two dimensional versions of the Alexander’s and Markov’s theorems on braids, by proving that any such a surface is isotopically equivalent to a closed braided surface and finding a set of moves relating isotopic braided surfaces.

In this paper we deal with the following question: can the above mentioned results be adapted in order to handle non-orientable surfaces in $S^4$, replacing the standard 2-sphere as base model for braided surfaces with some standard non-orientable surface, such as the Veronese surface (see section 3)?

The question is relevant in relation to the representation of orientable closed smooth 4-manifolds as branched covers of $S^4$, in which non-orientable surfaces play an essential role as branch sets in $S^4$ (see [6] and [17]).
Unfortunately, the answer is generally negative, in spite of some partial result obtained by Kamada (cf. [7]). In fact, in section 3 we show that there are very restrictive conditions for a non-orientable smooth surface in $S^4$ to be isotopic to a braided one. Nevertheless, we don’t know whether any orientable smooth closed 4-manifold is a cover of $S^4$ branched over a (possibly non-orientable) braided surface.

In order to study non-orientable braided surfaces in $S^4$, in section 2 we consider braided surfaces in $R^2$-bundles over surfaces and prove a few of preliminary results about them, which are of some interest independently of the present application.

This paper is a revised version of part of the degree thesis [21] written by the second author under the supervision of the first author.

1. Preliminaries

To begin with, we reformulate in terms of coverings the classical Artin’s notion of braid. By a geometric braid of degree $d$ in $R^3$ we mean a 1-submanifold $b \subset [0, 1] \times R^2 \subset R^3$ such that the canonical projection $\pi : [0, 1] \times R^2 \to [0, 1]$ restricts to a covering $\pi_b : b \to [0, 1]$ of degree $d$ and moreover, putting $b_i = \{x \in R^2 \mid (t, x) \in b\}$, we have $b_0 = b_1 = \ast$ for a fixed $\ast = \{\ast_1, \ldots, \ast_d\} \subset R^2$.

Considering braids of degree $d$ up to fibre preserving (with respect to $\pi$) ambient isotopy of $[0, 1] \times R^2$, we can think of them as elements of the braid group $B_d = \pi_1(S_dR^2, \ast)$ of degree $d$, where $S_dR^2 \cong (\Pi_dR^2 - \Delta)/\Sigma_d$ denotes the space of all the subsets of $R^2$ consisting of $d$ distinct points.

We recall the standard presentation of $B_d$ (cf. [4]), with generators $x_1, \ldots, x_{d-1}$, defined as shown in figure 1, and relations $x_ix_j = x_jx_i$ for any $i, j = 1, \ldots, d - 1$ such that $|i - j| > 1$ and $x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$ for any $i = 1, \ldots, d - 2$.

![Figure 1](image)

Given a braid $b = x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k} \in B_d$, we define the index of $b$ to be the exponent sum $i(b) = \varepsilon_1 + \ldots + \varepsilon_k$. Since all the relations above are balanced, it immediately follows that $i(b)$ is well-defined, that is it does not depend on the particular expression of $b$ as a power product of standard generators.

We call a closed braid of degree $d$ in $R^3$ any link $l \subset N(S^1) \subset R^3$, where $N(S^1)$ is a fixed open tubular neighborhood of $S^1$ in $R^3$, such that the orthogonal projection $\pi : N(S^1) \cong S^1 \times R^2 \to S^1$ restricts to a covering $\pi|_l : l \to S^1$ of degree $d$. By Alexander’s theorem, any link in $R^3$ is ambient isotopic to a closed braid.
The closure of a braid $b \in B_d$ is the closed braid $\widehat{b} = (\varphi \times \text{id}_{B^2})(b)$, where $\varphi : [0, 1] \to S^1$ is the usual parametrization given by $\varphi(t) = (\cos 2\pi t, \sin 2\pi t)$. Of course, $\widehat{b}$ is defined only up to fibre preserving ambient isotopy of $N(S^1) \cong S^1 \times \mathbb{R}^2$, being $b \in B_d$ defined only up to fiber preserving ambient isotopy of $[0, 1] \times \mathbb{R}^2$. Vice versa, $\widehat{b}$ uniquely determines $b$ up to conjugation in $B_d$.

Then, it makes sense to define the index of a closed braid $l$ in $\mathbb{R}^3$ by putting $i(l) = i(b)$, where $b \in B_d$ is any braid such that $l = \widehat{b}$, being the index of braids obviously invariant under conjugation in $B_d$.

The index $i(l)$ of a closed braid $l$ of degree $d$ satisfies the following Bennequin inequality (cf. [3]), involving the Euler characteristic $\chi(S)$ of any surface $S \subset \mathbb{R}^3$ such that $l = \text{Bd} S$ (that is a Seifert surface for $l$): $|i(l)| \leq d - \chi(S)$.

Finally, we recall the notion of branched covering between surfaces, which is needed in order to consider braided surfaces. A map $p : S \to X$ between compact surfaces is called a branched covering iff at any $s \in S$ it is locally equivalent to the complex map $z \mapsto z^{d(s)}$, where $d(s) \geq 1$ is the local degree of $p$ at $s$. The branch points of $p$ are the images of the singular ones, that is of the points $s \in S$ such that $d(s) > 1$. Moreover, $p$ a called simple if $d(s) = 2$ for any singular point $s \in S$ and $p$ is injective on the singular points.

2. Braided surfaces in fiber bundles

Let $f : N \to X$ be an $\mathbb{R}^2$-bundle over a compact connected surface $X$ with (possibly empty) boundary. We call (simple) braided surface of degree $d$ over $X$ any locally flat compact surface $S \subset N$ such that the restriction $p = f|_S : S \to X$ is a (simple) branched covering of degree $d$. Moreover, we call twist point of $S$ any singular point $t \in S$ of $p$ and denote by $d(t) \geq 2$ the local degree of $S$ at $t$, that is the local degree of $p$ at $t$.

For any twist point $t \in S$, there exists a commutative diagram like the following, where: $C \subset N$ is a closed neighborhood of $t$, $D \subset X$ is a closed neighborhood of $p(t)$, $h$ and $k$ are homeomorphisms, $b_t \subset S^1 \times \text{Int} B^2$ is a closed braid of degree $d(t)$, $C(b_t) \subset B^2 \times B^2$ is the cone of $b_t$ with vertex $(0,0)$, $\pi$ is the canonical projection on the first factor.

\[
\begin{array}{ccc}
  t & \in & S \cap C \\
  \downarrow & & \downarrow h \\
(0,0) & \in & C(b_t) \subset B^2 \times B^2 \\
  \downarrow & & \downarrow k \\
  & \overset{f|_C}{\longrightarrow} & D \\
  \end{array}
\]

If $N$ is oriented, we can assume that $h$ is orientation preserving (with respect to the standard orientation of $B^2 \times B^2$). Moreover, fixed any local orientation of $X$ at $p(t)$, we can also assume that $k$ is orientation preserving (with respect to the standard orientation of $B^2$). With these two assumptions, $b_t$ turns out to be uniquely determined up to braid isotopy, in such a way that we can define the local index $i(t)$ of $S$ at $t$ to be the integer number $i(b_t)$. In fact, it can be easily seen that $i(t)$ depends only on $S$ and on the orientation of $N$, while it does not depend on the choice of the local orientation of $X$. 

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If $t$ is a simple twist point of $S$, then, by local flatness, $b_t$ coincides with the closure of one of the braids $x_{i}^{\pm 1} \in B_2$, so that $C(b_t)$ can be thought to have equation $w = z^2$ or $w = \bar{z}^2$ (depending on the sign of the exponent), with respect to the complex coordinates $(w,z)$ of $B^2 \times B^2 \subset C^2$.

On the other hand, if $t$ is a smooth twist point of $S$, then we get for $C(b_t)$ the equation $w = z^{d(t)}$ or $w = \bar{z}^{d(t)}$, while $b_t$ turns out to be the closure of the braid $(x_1 \cdots x_{d(t)-1})^{\pm 1} \in B_{d(t)}$.

Hence, any simple twist point of $S$ is smooth (up to fiber preserving ambient isotopy of $N$), and any smooth twist point $t$ can be easily perturbed to get $d(t) - 1$ simple ones (up to ambient isotopy of $N$ which does not preserve the fibers of $f$). In this case, we can assign to the twist point $t \in S$ a sign $s(t) = \pm 1$, depending only on the local shape of $S$ and on the orientation of $N$, in such a way that $i(t) = s(t)(d(t) - 1)$.

For a non-smooth twist point $t$ it may not exist any simple perturbation up to ambient isotopy, as it is shown in [10]. Nevertheless, we can modify $S$ in a neighborhood of $t$ in order to get a new braided surface $S'$, where the twist point $t$ is replaced by a certain number of simple twist points $t_1, \ldots, t_k$ such that $i(t) = i(t_1) + \cdots + i(t_k)$. Namely, if $b_t$ is the closure of a braid $x_{j_1}^{\varepsilon_1} \cdots x_{j_k}^{\varepsilon_k} \in B_{d(t)}$, then we can replace $C(b_t)$ by a braided surface in $B^2 \times B^2$, having a positive (resp. negative) simple twist point $t_i$ for each $\varepsilon_i = +1$ (resp. $\varepsilon_i = -1$), with $l = 1, \ldots, k$ (cf. proposition 1.11 of [19]).

We remark that the braided surface we put in place of $C(b_t)$ is not necessarily a disk, however it is homologous to $C(b_t)$ mod the common boundary $b_t$. Then, $S$ and $S'$ may not be isotopic, but they share some homological properties, in particular they have the same self-intersection number as multi-valued sections of $f$. Moreover, by local flatness, the closed braid $b_t$ represents the unknot, hence the Bennequin inequality implies that $|i(t)| \leq d(t) - 1$.

Still assuming $N$ oriented, we define the index of $S$ as the sum $i(S) = \sum i(t)$, where $t$ runs over all the twist points of $S$. The following proposition gives the index of $S$ in terms of its degree $d$ and of the Euler number $e$ of the bundle $f : N \to X$ over a closed surface $X$.

**Proposition 2.1.** If $N$ is oriented and $S \subset N$ is any braided surface over a closed surface $X$ as above, then $i(S) = ed(d - 1)$.

**Proof.** By the definition of index and the previous observations, we can assume that $S$ is simple, so that $i(S)$ equals the algebraic number of the twist points of $S$.

To begin with, we consider a handlebody decomposition of the base surface $X$, consisting of one 0-handle $H^0$, 1-handles $H^1_1, \ldots, H^1_{2g+h}$ attached to $H^0$ in the standard way depicted in figure 2, with $g, h \geq 0$ and $H^1_j$ orientable (resp. non-orientable) for $j = 1, \ldots, 2g$ (resp. $j = 2g + 1, \ldots, 2g + h$), and one 2-handle $H^2$.

We can assume that all the branch points of $p$ belong to $\operatorname{Int} H^2$, in such a way that, putting $X_1 = H^0 \cup H^1_1 \cup \cdots \cup H^1_{2g+h}$, $N_1 = f^{-1}(X_1)$ and $S_1 = S \cap N_1$, the restriction $p|_{S_1} : S_1 \to X_1$ is an ordinary covering.

By a suitable choice of the trivializations of $f$ over the handles $H^0$ and $H^1$, we can think of $N_1$ as the quotient space obtained by attaching $H^1_j \times R^2$ to $H^0 \times R^2$ for all $j = 1, \ldots, 2g + h$, by fiber preserving maps, whose restrictions to the fibers coincide with $\operatorname{id}_{R^2}$ or $\sigma$, where $\sigma : R^2 \to R^2$ is the symmetry with respect to the $y$-axis.
Moreover, we can assume that the trivialization \( f^{-1}(H^0) \cong H^0 \times R^2 \), makes 
\( S_0 = S \cap f^{-1}(H^0) \into H^0 \times \{*_1, \ldots, *_d\} \subset H^0 \times R^2 \), where \(*_1, \ldots, *_d\) belong to
the \( x \)-axis and \( \sigma(\{*_1, \ldots, *_d\}) = \{*_1, \ldots, *_d\} \). Then, for every \( j = 1, \ldots, 2g + h \), the
trivialization \( f^{-1}(H^1_j) \cong H^1_j \times R^2 \) makes \( S \cap f^{-1}(C_j) \) into a braid \( c_j \subset C_j \times R^2 \),
where \( C_j \) denote the core of the handle \( H_j \), oriented as in figure 2.

We observe that \( S_1 \) is completely determined (up to fiber preserving iso-
topy) by the braids \( c_1, \ldots, c_{2g+h} \in B_d \) and \( BdS_1 \) is a closed braid in \( BdN_1 \cong Bd\pi_1 \times R^2 \cong S^1 \times R^2 \), which can be thought as the closure of the braid 
\( c = c_1c_2c_1^{-1}c_2^{-1} \cdots c_{2g-1}c_{2g}c_{2g-1}^{-1}c_{2g}^{-1}c_{2g+1}c_{2g+1}^{-1} \cdots c_{2g+h}c_{2g+h}^{-1} \in B_d \), where \( c_j \) denotes
the image of \( c_j \) under the action of \( \sigma \).

Putting \( N_2 = f^{-1}(H^2) \) and \( N_2 = S \cap N_2 \), we have that \( Bd(S_2) = Bd(S_1) \)
is a closed braid in \( BdN_2 \cong BdH^2 \times R^2 \cong S^1 \times R^2 \), which can be thought as the closure of the braid \( c' = ct^e \),
where \( t \in Bd \) denotes one positive full twist of \( d \) strings. On the other hand, denoting by \( t_1, \ldots, t_n \in S_2 \) the (simple) twist
points of \( S \), it is straightforward (for example, see proposition 1.11 of [19]) to get 
\( c' = y_1x_j^{s(t_1)}y_1^{-1} \cdots y_nx_j^{s(t_n)}y_n^{-1} \), where each \( x_j \) is a standard generator of \( B_d \) and
\( s(t_j) = \pm 1 \) as above.

At this point, we can finish the proof by observing that the computation of \( i(c') \)
based on the first expression of \( c' \) as a product of powers of generators gives us
\( ed(d-1) \), while the second one gives us \( s(t_1) + \ldots + s(t_k) \), that is \( i(S) \). \( \square \)

As a consequence of proposition 2.1, we get numerical obstructions to the exist-
ence of braided surfaces, in terms of Euler characteristic and number of twist
points.

**Proposition 2.2.** If \( N \) is oriented and \( S \subset N \) is any braided surface over a
closed surface \( X \) as above, then \( \chi(S) \leq d(\chi(X) - |e|(d-1)) \). Moreover, if \( S \) is simple,
then the number of twist points is even and not less than \( |e|d(d-1) \).

**Proof.** By the Hurwitz formula we have \( \chi(S) = d\chi(X) - \sum(d(t) - 1) \). Then,
the first part of the proposition follows immediately by proposition 2.1 and by the
inequalities \(|i(S)| \leq \sum|i(t)| \leq \sum(d(t) - 1) \). For the second part, it is enough
to observe that, if \( S \) is simple, then the number of twist points coincides with
\( \sum(d(t) - 1) \), which is congruent to \( i(S) = \sum s(t)(d(t) - 1) \mod 2 \). \( \square \)

Now, we want to show that the inequalities given by the proposition 2.2 are
sharp. Given any \( R^2 \)-bundle \( f : N \to X \) with oriented total space \( N \) and arbitrary
Euler number $e > 0$ (the case $e < 0$ can be covered by reversing the orientation of $N$), let $S_1, \ldots, S_d \subset N$ be $d$ smooth sections of $f$, transversally meeting each other in $e$ points. Then, replacing each of the $ed(d - 1)/2$ double points of $S_1 \cup \ldots \cup S_d$ with one pair of positive simple twist points, as shown in figure 3, we get a simple braided surface $S$ of degree $d$ over $X$, with $ed(d - 1)$ positive twist points and $\chi(S) = d(\chi(X) - e(d - 1))$. On the other hand, we can easily add to $S$ pairs of opposite simple twist points, as shown in figure 4, in order to arbitrarily increase the number of twist points of $S$ and decrease the Euler characteristic $\chi(S)$.

![Figure 3.](image1)

![Figure 4.](image2)

We conclude this section by computing the Euler number $e(S)$ of the braided surface $S$, that is the self-intersection number of $S$ in the oriented 4-manifold $N$, which coincides with the self-intersection of $S$ as a multi-valued section of $f$.

**Proposition 2.3.** If $N$ is oriented and $S \subset N$ is any braided surface over a closed surface $X$ as above, then $e(S) = i(S) + ed = ed^2$.

*Proof.* Let $s : X \to N$ be a cross section of $f$ transverse to the null section. We can assume that $S$ is simple and that the zeroes of $s$ are not branch points. By translating $s(x)$ at every point in $S \cap f^{-1}(x)$ for every $x \in X$ and taking normal component with respect to $S$, we get a normal vector field $v$ along $S$ with non-degenerate singularities.

A point $y \in S$ is a singular point for $v$ iff $f(y)$ is a singular point for $s$ or $y$ is a twist point for $S$; furthermore all the signs are coherent. Therefore we have $e(S) = i(S) + ed$. Then, the statement follows by proposition 2.1. □

We notice that the results above can be easily generalized to the case of singular braided surfaces with transversal double points, by taking account of each double point as a pair of twist points. Namely, denoting by $n(S)$ the algebraic number of double points of $S$, we have $i(S) + 2n(S) = ed(d - 1)$ and $e(S) = ed^2 - 2n(S)$. 
3. Non-orientable braided surfaces in $S^4$

In this section we apply the results of the previous one, in order to show that the Viro-Kamada’s representation theorem of orientable surfaces in $S^4$ as braided surfaces (cf. [8]) cannot be extended to include the non-orientable case.

In fact, by combining the results of the previous section with the Whitney’s conjecture on non-orientable surfaces in $S^4$, proved by Massey in [14], we get very restrictive conditions for such a surface to be isotopic to a braided one, with respect to any reasonable definition of non-orientable braided surface. We recall that the Whitney conjecture imposes the following constrains to the self-intersection number $e$ of a non-orientable surface of Euler characteristic $\chi$ in $S^4$: $e \equiv 2\chi \mod 4$ and $|e| \leq 4 - 2\chi$.

It is natural to call a non-orientable braided surface in $S^4$ any non-orientable surface $S \subset S^4$ which is contained as a braided surface over $X$ in the normal fiber bundle $\nu : N \to X$ of some fixed standard smooth non-orientable surface $X \subset S^4$, where $N$ is identified with an open tubular neighborhood of $X$ in $S^4$.

The most significant choice for $X$ is the Veronese surface $V \subset S^4$ defined in the following way. First of all, we consider the space $\mathcal{M} \simeq R^9$ of the $3 \times 3$ matrices over $R$ with the inner product given by $\langle A, B \rangle = \text{tr}(AB^T)$ for all $A, B \in \mathcal{M}$, and the map $\varphi : S^2 \to \mathcal{M}$ defined by $\varphi(x) = x^T x$ for any $x \in S^2 \subset R^3$. Since $\varphi(y) = \varphi(x)$ iff $y = \pm x$, we get an induced embedding $\psi : P^2 \to \mathcal{M}$, where $P^2$ is thought as the quotient of $S^2$ by the action of the antipodal map $x \mapsto -x$. Then, we put $V = \psi(P^2)$ after having identified $S^4$ with the intersection of the unit sphere of $\mathcal{M}$ with the affine subspace $L = \{ M \in \mathcal{M} \mid M = M^T \text{ and } \text{tr} \, M = 1 \} \simeq R^5$.

The remarkable property that characterizes $V$ is the existence of a symmetric splitting $S^4 \simeq \overline{N} \cup_f \overline{N}$, where $\overline{N}$ is a closed tubular neighborhood of $V$ in $S^4$ and $f$ is an involution of $\text{Bd} \, \overline{N}$ onto itself (see [13] and [16]). Such splitting has several relevant geometric properties (cf. [1] and [16]), moreover, from a topological point of view, it is related to the identification of $S^4$ with the quotient of the complex projective plane under complex conjugation, being $V$ the branch set of the canonical projection $CP^2 \to CP^2 / \zeta \simeq S^4$ (cf. [12], [13] and [15]).

**Corollary 3.1.** Any non-orientable braided surface $S \subset S^4$ of degree $d$ over the Veronese surface $V \subset S^4$ satisfies the following conditions: $\chi(S) \leq d(3 - 2d)$, $i(S) = 2d(d - 1)$ and $e(S) = 2d^2$. As a consequence, the only surface $S$ braided over $V$ with $\chi(S) = 1$ is $V$ itself (up to isotopy of braided surfaces) and there is no surface $S$ braided over $V$ with $\chi(S) = 0, -1, -3, -5, -7$.

**Proof.** The first part of the corollary immediately follows from the results of the previous section, by taking into account that $e(V) = 2$. Now, the first inequality implies that: for $\chi(S) = 1$ we have $d = 1$, that is $S \simeq V$; on the other hand, for $d \geq 2$ we have $\chi(S) \leq -2$; furthermore, we have $d \geq 3$, that gives us $\chi(S) \leq -9$, if we assume $\chi(S)$ odd and less than 1, since in this case also $d$ is odd, because of the congruence $2d^2 \equiv 2\chi(S) \mod 4$. □

We remark that, by the equation $e(S) = 2d^2$, any surface $S \subset S^4$ braided over $V$ has positive self-intersection. In order to get negative (resp. vanishing) self-intersection numbers, one could consider surfaces braided over $V' = \alpha(V)$ (resp. $V \# V'$), where $\alpha : S^4 \to S^4$ is the antipodal map.

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Moreover, it is worth observing that, in the non-vanishing cases, only few values of the self-intersection among the ones allowed by the Whitney conjecture are realized by surfaces braided over \( V \) or \( V' \). In fact, the self-intersection number of such a surface, besides having the very special form \( e(S) = \pm 2d^2 \), is bounded by the inequality \(|e(S)| \leq 9/4 - \chi(S) + 3/4 \sqrt{9 - 8\chi(S)}\), that can be derived from the inequality of corollary 3.1 by a straightforward computation.

However, the following problem remains still open: *is it possible to represent any orientable closed smooth 4-manifold as a cover of \( S^4 \) branched over a (possibly non-orientable) braided surface?*

**References**


