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AUTOMORPHISMS OF TRIVALENT GRAPHS

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Abstract

Let $\mathcal{G}_{g,b}$ be the set of all uni/trivalent graphs representing the combinatorial structures of pant decompositions of the oriented surface $\Sigma_{g,b}$ of genus g with b boundary components. We describe the set $\mathcal{A}_{g,b}$ of all automorphisms of graphs in $\mathcal{G}_{g,b}$ showing that, up to suitable moves changing the graph within $\mathcal{G}_{g,b}$, any such automorphism can be reduced to elementary switches of adjacent edges.

Keywords: uni/trivalent graph, F-move, pant decomposition complex.

AMS Classification: Primary 20B25; Secondary 57M50, 05C25.

Introduction

Let $\mathcal{G}_{g,b}$ be the set of all connected non-oriented uni/trivalent graphs Γ, with first Betti number $\beta_1(\Gamma) = g \geq 0$, at least one trivalent vertex and $b \geq 0$ univalent vertices. It is immediate to see that such a graph Γ exists if and only if $(g, b) \neq (0,0), (0,1), (0,2), (1,0)$, and in this case Γ has $2g - 2 + b$ trivalent vertices and $3g - 3 + 2b$ edges. In the following, the pair (g, b) will be always assumed to satisfy that restriction.

Given any $\Gamma \in \mathcal{G}_{g,b}$, we denote by $\mathcal{A}(\Gamma)$ the group of all the automorphisms of Γ as a non-oriented graph. In this paper we are interested in the structure of the set $\mathcal{A}_{g,b} = \cup_{\Gamma \in \mathcal{G}_{g,b}} \mathcal{A}(\Gamma)$ of all the automorphisms of graphs in $\mathcal{G}_{g,b}$. More precisely, we will prove that, up to certain F-moves changing the graph within $\mathcal{G}_{g,b}$, any such automorphism can be reduced to elementary switches of adjacent edges.

The drive for this paper comes from a problem we met while working on pant decompositions of surfaces in [3], that is described in the Appendix of the present paper. However, the study of the automorphisms of uni/trivalent graphs has its own interest independently from that specific application. This is somehow proven by the rather extensive literature on the theme, which starts from the classical papers [15, 16] and [11], and involves some more recent ones, as for instance [6, 5, 8, 9]. Hence, in this paper we focus on the structure of $\mathcal{A}_{g,b}$, while the application to pant decompositions will be described in detail in a forthcoming paper [4].

Moreover, we emphasize that our result may be easily restated in term of rotors, i.e. subgraph of G having rotational symmetry, and whose vertices of attachment are equivalent with respect to this symmetry. Rotors were introduced in graph theory by Tutte in [17], and the concept was adapted to knot theory as a generalization of Conway's mutation (see for instance [1] and [7]). A careful analysis of those applications, in the light of our results, will be the object of further investigations.

The paper is structured as follows. In Section 1 we give the basic definitions, state our main result as Theorem 1.9, and outline its proof by reducing it to the special case of prime power order automorphisms. Then, after having established some preliminary results in Section 2, we provide the fundamental ingredients for the proof of Theorem 1.9 in the subsequent Sections 3, 4, 5. More precisely, in Sections 3 and 4 we show that an automorphism in $\mathcal{A}_{g,b}$ is F-equivalent to the composition of two automorphisms of order 2, if it has order p^m with $p \geq 3$ or 3^m respectively. Finally, the case of automorphisms of order 2^m is considered in Section 5.

1. Definitions and main theorem

For a graph $\Gamma \in \mathcal{G}_{g,b}$, we call a *free end* of Γ any univalent vertex, a *terminal edge* of Γ any of the b edges connecting a free end to a trivalent vertex, and an *internal edge* of Γ any of the $3g-3+b$ edges connecting two (possibly coinciding) trivalent vertices.

First of all, we introduce F-moves on uni/trivalent graphs. These are well-known moves that change the graph structure by acting on the internal edges. It is a folklore result that F-moves, even in the most restrictive form given in Definition 1.1, suffice to relate any two graphs in $\mathcal{G}_{g,b}$ up to graph isomorphisms (cf. [13]). However, we need to consider an invariant version of F-moves, in order to relate graph automorphisms. This makes things more involved.

Definition 1.1. Let $\Gamma \in \mathcal{G}_{g,b}$ and $e \in \Gamma$ be an internal edge with distinct ends. Then, we call *elementary (edge) F-move*, the modification $F_{e,e'} : \Gamma \rightsquigarrow \Gamma'$ that makes Γ into $\Gamma' \in \mathcal{G}_{g,b}$, by replacing e with a new internal edge e' as in Figure 1, while leaving the rest of the graph unchanged. Clearly, the inverse modification $F_{e',e} : \Gamma' \rightsquigarrow \Gamma$ is an elementary (edge) F-move as well.

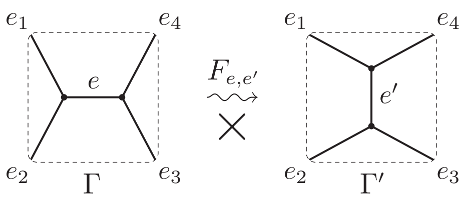

Figure 1. An edge F-move $F_{e,e'}$.

According to the literature, here the letter F stands for "fusion" and refers to the fact that the F-move $F_{e,e'}$ can be thought as the contraction of the edge e in Γ, and the consequent fusion of its ends, followed by the inverse of a similar contraction of the edge e' in Γ'. The intermediate graph has a quadrivalent vertex in place of the two trivalent ends of the contracted edges, as indicated under the arrow.

We warn the reader that, since the graphs in $\mathcal{G}_{g,b}$ are abstract ones, properties and constraints depending on the planar representation (of portions) of them in the figures do not have any significance.

In particular, the coupling $(e_1e_2)(e_3e_4)$ of the (possibly not all distinct) edges $e_1, \dots, e_4$ determined by the two ends of e in Γ, which in Figure 1 is replaced by the coupling $(e_1e_4)(e_2e_3)$ determined by the two ends of e' in Γ', could be replaced by the coupling $(e_1e_3)(e_2e_4)$ as well. In other words, there are exactly two possible ways to perform the F-move $F_{e,e'}$ at the given edge e of Γ, corresponding to the two possible changes of coupling, depending on the structure of Γ' at the edge e':

$$(e_1e_2)(e_3e_4) \overset{F_{e,e'}}{\rightsquigarrow} (e_1e_4)(e_2e_3) \quad \text{and} \quad (e_1e_2)(e_3e_4) \overset{F_{e,e'}}{\rightsquigarrow} (e_1e_3)(e_2e_4).$$

From a different perspective, the elementary F-move described in Figure 1, can be interpreted as the replacement of a uni/trivalent subtree $T \subset \Gamma$ with a different uni/trivalent subtree $T' \subset \Gamma'$ having the same univalent vertices. Namely, T consists of the edges $e_1, \dots, e_4$ and e, while T' consists of the edges $e_1, \dots, e_4$ and e'. This suggests the following generalization of the notion of elementary F-move.

Definition 1.2. Let $\Gamma \in \mathcal{G}_{g,b}$ and $T \subset G$ be a uni/trivalent subtree with $m \geq 4$ free ends. Then, we call *elementary (tree) F-move*, the modification $F_{T,T'} : \Gamma \rightsquigarrow \Gamma'$ that makes Γ into $\Gamma' \in \mathcal{G}_{g,b}$, by replacing T with a different uni/trivalent tree $T' \subset \Gamma'$ having the same free ends of T, while leaving the rest of the graph unchanged (see Figure 2 for an example with $m = 5$). Also in this case, the inverse modification $F_{T',T} : \Gamma' \rightsquigarrow \Gamma$ is an elementary (tree) F-move as well.

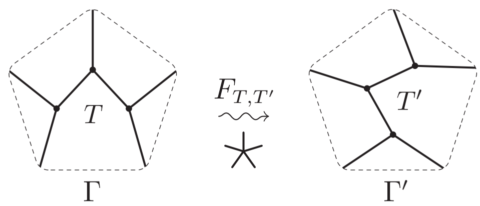

FIGURE 2. A tree F-move $F_{T,T'}$.

We note that the F-move $F_{T,T'}$ is again a kind of fusion move. In fact, it can be thought as the contraction of the subtree consisting of all the internal edges of $T \subset \Gamma$ to a single m-valent vertex, and the consequent fusion of all the trivalent vertices of T, followed by the inverse of a similar contraction for $T' \subset \Gamma'$.

If the subtree $T \subset \Gamma$ has the vertices $v_1, \dots, v_m$ as its free ends, then all the possible ways to perform the F-moves $F_{T,T'}$ on Γ corresponds to all the complete iterated couplings of the (unordered) set $\{v_1, \dots, v_m\}$ induced by T', except the one induced by T. In particular, according to the above discussion, we have 2 ways for $m = 4$.

Actually, any tree F-move $F_{T,T'}$ can be realized by a suitable sequence of edge F-moves performed on the edges of T (and of the corresponding subtrees in the resulting graphs). Yet, it makes sense to consider $F_{T,T'}$ as a unique move, and we will see shortly why (cf. discussion about the F-move in Figure 3 below).

Definition 1.3. Let $\Gamma \in \mathcal{G}_{g,b}$ and $\mathcal{T} = \{T_1, \dots, T_k\}$ be a family of uni/trivalent subtrees of Γ with pairwise disjoint interiors (that is, two of them possibly share only some common free ends). Assuming that each T_i has at least 4 free ends, we call *F-move* any modification $F_{\mathcal{T},\mathcal{T}'} : \Gamma \rightsquigarrow \Gamma'$ given by the simultaneous application of certain elementary moves $F_{T_1,T_1'}, \dots, F_{T_k,T_k'}$ to Γ. Of course, for $k = 1$ we have an elementary F-move.

For any graph $\Gamma \in \mathcal{G}_{g,b}$ we denote by $\mathcal{A}(\Gamma)$ the group of all the automorphisms of Γ as a non-oriented graph, including those that permute the free ends. Moreover, we denote by $\mathcal{A}_{g,b} = \cup_{\Gamma \in \mathcal{G}_{g,b}} \mathcal{A}(\Gamma)$ the set of all the automorphisms of graphs in $\mathcal{G}_{g,b}$.

Now, assume we are given an automorphism $\varphi \in \mathcal{A}(\Gamma)$. If $F_{\mathcal{T},\mathcal{T}'} : \Gamma \rightsquigarrow \Gamma'$ is an F-move with $\mathcal{T} = \{T_1, \dots, T_k\}$ a φ-invariant family of subtrees of Γ, then the restriction of φ to $\Gamma - \cup_i \operatorname{Int} T_i$ induces a graph automorphism ψ of $\Gamma' - \cup_i \operatorname{Int} T_i'$, under the canonical isomorphism $\Gamma - \cup_i \operatorname{Int} T_i \cong \Gamma' - \cup_i \operatorname{Int} T_i'$ of graphs given by $F_{\mathcal{T},\mathcal{T}'}$. In general, such ψ does not extend to an automorphism $\varphi' \in \mathcal{A}(\Gamma')$, but if it does then the extension $\varphi' : \Gamma' \to \Gamma'$ can be easily seen to be unique, by taking into account that $\Gamma - \cup_i \operatorname{Int} T_i$ contains all the free ends of all the subtrees T_i. When φ' exists, we say that it is induced by φ through $F_{\mathcal{T},\mathcal{T}'}$.

Definition 1.4. Given an automorphism $\varphi : \Gamma \to \Gamma$, we say that an F-move $F_{\mathcal{T},\mathcal{T}'} : \Gamma \rightsquigarrow \Gamma'$ is *φ-invariant* if $\mathcal{T}$ is a φ-invariant family of subtrees and φ induces an automorphism $\varphi' : \Gamma' \to \Gamma'$ through $F_{\mathcal{T},\mathcal{T}'}$ as discussed above. In this case, we write $F_{\mathcal{T},\mathcal{T}'} : \varphi \rightsquigarrow \varphi'$. Such relation between automorphisms is symmetric, in the sense that $F_{\mathcal{T}',\mathcal{T}} : \varphi' \rightsquigarrow \varphi$ holds as well, being $F_{\mathcal{T}',\mathcal{T}} : \Gamma' \rightsquigarrow \Gamma$ a φ'-invariant F- move. But it is neither reflexive nor transitive. Then, we call *F-equivalence* the generated equivalence relation on the set $\mathcal{A}_{g,b}$.

We observe that the elementary F-move in Figure 1 is φ-invariant with respect to any automorphism $\varphi \in \mathcal{A}(\Gamma)$ that acts on the depicted portion of Γ as any planar symmetry (horizontal, vertical or central). On the contrary, it is not φ-invariant for a φ that switches e_1 and e_2 while leaving e_3 and e_4 fixed.

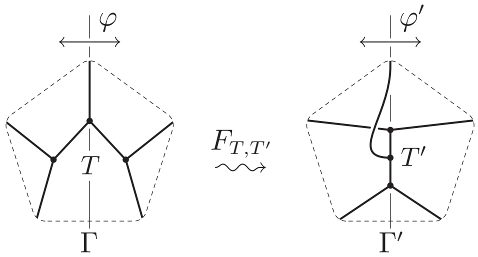

FIGURE 3. A φ-invariant F-move.

Figure 3 shows a φ-invariant elementary F-move $F_{T,T'}$, where $\varphi \in \mathcal{A}(\Gamma)$ is any automorphism that acts on T as the simmetry with respect to the vertical axis. As we said above, $F_{T,T'}$ could be realized by a sequence of elementary edge F-moves, but this is not true if we insist that each single edge F-move is φ-invariant. Finally, notice that instead the F-move described in Figure 2 is not φ-invariant (for the same automorphism φ).

Remark 1.5. In the following, we will essentially use only φ-invariant F-moves $F_{\mathcal{T},\mathcal{T}'}$ consisting of simultaneous elementary edge F-moves $F_{e,e'}$ or tree F-moves $F_{T,T'}$ like the one in Figure 3. We emphasize once again that the φ-invariance of F-moves $F_{\mathcal{T},\mathcal{T}'}$ of does not imply the φ-invariance of each single elementary F-move, being these possibly permuted by φ. In many cases, it will be convenient to think of those elementary moves as the edges slidings shown in Figure 4. Here, the edges f, f_1 and f_2 are the original ones of Γ, whose ends in the figure are slided to get their new positions in Γ' as indicated by the arrows, while the rest of the graph is fixed.

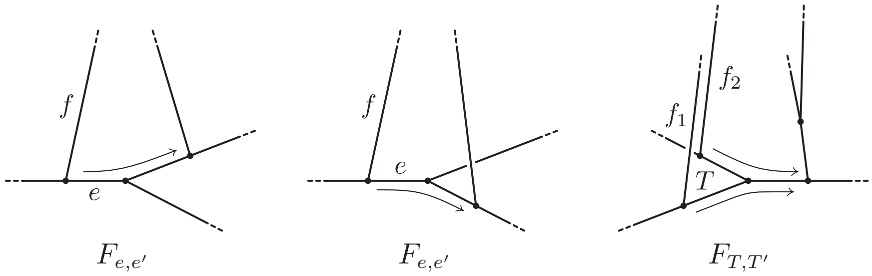

FIGURE 4. F-moves as edge slidings.

At this point, in order to state our main theorem, we are left to introduce the elementary automorphisms, which will generate all the automorphisms in $\mathcal{A}_{g,b}$ up to F-equivalence.

Definition 1.6. By an *elementary automorphism* of a graph $\Gamma \in \mathcal{G}_{g,b}$ we mean any automorphism $S_{e_1,e_2} : \Gamma \to \Gamma$ interchanging two adjacent edges e_1 and e_2 of Γ, while fixing all the rest of the graph. We call such an automorphism $S_{e_1,e_2} \in \mathcal{A}(\Gamma)$ a *terminal switch* or an *internal switch*, depending on the fact that e_1 and e_2 are terminal or internal edges (cf. Figure 5).

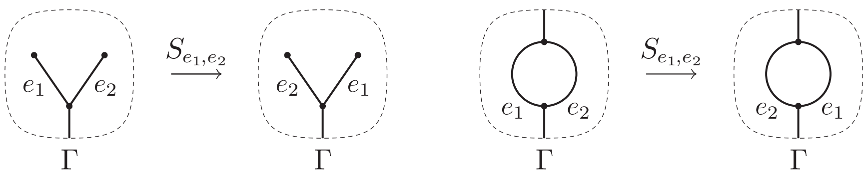

FIGURE 5. A terminal switch (left side) and an internal switch (right side).

Remark 1.7. An automorphism in $\Gamma \in \mathcal{G}_{g,b}$ is uniquely determined up to internal switches by its action on the vertices of Γ. Moreover, it is uniquely determined up to (terminal and internal) switches by its action on the trivalent vertices of Γ.

To simplify our claims, we provide the next definition.

Definition 1.8. For any g and b, we denote by $\mathcal{E}_{g,b} \subset \mathcal{A}_{g,b}$ the smallest subset that contains all the elementary automorphisms (internal and terminal switches) in $\mathcal{A}_{g,b}$ and is closed with respect to composition and F-equivalence of automorphisms.

Then, our main result can be stated as follows.

Theorem 1.9. *For any g and b, we have $\mathcal{E}_{g,b} = \mathcal{A}_{g,b}$.*

Proof. We have to show that any element $\varphi \in \mathcal{A}_{g,b}$, that is any automorphism $\varphi : \Gamma \to \Gamma$ of a graph $\Gamma \in \mathcal{G}_{g,b}$, actually belongs to $\mathcal{E}_{g,b}$. By primary decomposition of the cyclic subgroup $\langle\varphi\rangle \subset \mathcal{A}(\Gamma)$, we can reduce ourselves to the special case when the order $\operatorname{ord}(\varphi)$, that is the cardinality of $\langle\varphi\rangle$, is a prime power. Then, the subcases when $\operatorname{ord}(\varphi) = p^m$ with p prime > 3 or $\operatorname{ord}(\varphi) = 3^m$ are reduced to the case $\operatorname{ord}(\varphi) = 2$ in Propositions 3.3 and 4.3 respectively. Finally, the case of $\operatorname{ord}(\varphi) = 2^m$ follows from Propositions 5.4 and 5.5. $\square$

2. Some preliminary results

Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a uni/trivalent graph Γ. Two elements (vertices or edges) or subgraphs x and y of Γ are said to be *φ-equivalent* if $y = \varphi^i(x)$ for some $i \geq 0$. In this case we write $x \cong_\varphi y$.

We denote by $\operatorname{ord}(\varphi)$ the order of φ in $\mathcal{A}(\Gamma)$ (as an automorphism of a non-oriented graph), that is the cardinality of $\langle\varphi\rangle \subset \mathcal{A}(\Gamma)$. For a vertex v of Γ, by the *order of v with respect to φ*, we mean as usual the cardinality $\operatorname{ord}_\varphi(v)$ of its φ-orbit $\operatorname{Orb}_\varphi(v) = \{\varphi^k(v) \,|\, 0 \leq k < \operatorname{ord}(\varphi)\}$. On the contrary, for an edge e of Γ with (possibly coinciding) ends v and w, we call *order of e with respect to φ* the number $\operatorname{ord}_\varphi(e) = \operatorname{LCM}(\operatorname{ord}_\varphi(v), \operatorname{ord}_\varphi(w))$, which is equal to either the cardinality of the φ-orbit $\operatorname{Orb}_\varphi(e) = \{\varphi^k(e) \,|\, 0 \leq k < \operatorname{ord}(\varphi)\}$ or its double (if $v \neq w$ and $(v, w) \cong_\varphi (w, v)$ as ordered pairs). This amounts to consider e as an oriented edge, except the case when it is a loop.

Lemma 2.1. *If φ and φ' are F-equivalent in $\mathcal{A}_{g,b}$, then* $\operatorname{ord}(\varphi) = \operatorname{ord}(\varphi')$.

Proof. It suffices to consider the case when φ and φ' are related by a single F-move $F_{\mathcal{T},\mathcal{T}'} : \varphi \rightsquigarrow \varphi'$, with $\mathcal{T} = \{T_1, \dots, T_k\}$ and $\mathcal{T}' = \{T'_1, \dots, T'_k\}$. Then, $\operatorname{ord}(\varphi)$ coincides with the order of its restriction to $\Gamma - \cup_i \operatorname{Int} T_i$, by the uniqueness of extension to the subtrees T_i. Similarly, $\operatorname{ord}(\varphi')$ coincides with the order of its restriction to $\Gamma' - \cup_i \operatorname{Int} T'_i$. Since those restrictions coincide under the canonical isomorphism $\Gamma - \cup_i \operatorname{Int} T_i \cong \Gamma' - \cup_i \operatorname{Int} T'_i$, we have $\operatorname{ord}(\varphi) = \operatorname{ord}(\varphi')$. $\square$

Lemma 2.2. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a (possibly disconnected) uni/trivalent graph Γ. If v is a vertex of Γ and e is an edge of Γ having v as an end, then we have one of the following:*

1) $\operatorname{ord}_\varphi(e) = \operatorname{ord}_\varphi(v)$;
2) $\operatorname{ord}_\varphi(e) = 2 \operatorname{ord}_\varphi(v)$, *which happens if and only if v is a trivalent vertex, with three distinct edges e, e' and e'' exiting from it, such that $e' = \varphi^{\operatorname{ord}_\varphi(v)}(e)$ and* $\operatorname{ord}_\varphi(e'') = \operatorname{ord}_\varphi(v)$;
3) $\operatorname{ord}_\varphi(e) = 3 \operatorname{ord}_\varphi(v)$, *which happens if and only if v is a trivalent vertex, with three distinct edges e, e' and e'' exiting from it, such that $e' = \varphi^{\operatorname{ord}_\varphi(v)}(e)$ and $e'' = \varphi^{2 \operatorname{ord}_\varphi(v)}(e)$.*

Moreover, if Γ is connected then the set of all orders of edges is $\{m, 2m, \dots, 2^k m\}$ for some $m \geq 1$ and $k \geq 0$ such that $\operatorname{ord}(\varphi) = 2^k m$, while the set of all orders of vertices is $\{m, 2m, \dots, 2^h m, 2^{j_1} m/3, \dots, 2^{j_\ell} m/3\}$ with $k - 1 \leq h \leq k$, $0 \leq l \leq k$, $0 \leq j_1 < \dots < j_\ell \leq k$, and m multiple of 3 if $\ell > 0$.

Proof. By definition of order for edges, we immediately have that $\operatorname{ord}_\varphi(e)$ is a multiple of $\operatorname{ord}_\varphi(v)$. Moreover, since $\varphi^{i \operatorname{ord}_\varphi(v)}(e)$ is an edge exiting from $\varphi^{i \operatorname{ord}_\varphi(v)}(v) =$

v for any $i \geq 1$, the only possible cases are $\mathrm{ord}_\varphi(e) = \mathrm{ord}_\varphi(v), 2\,\mathrm{ord}_\varphi(e), 3\,\mathrm{ord}_\varphi(v)$ and these are verified according to the conditions given in the statement. As a consequence, if the orders of two adjacent edges are different, then one of them is the double of the other, being those edges like e and e'' in point 2. For Γ connected, this implies that the set of orders of the edges is $\{m, 2m, \dots, 2^k m\}$ for some $m \geq 1$ and $k \geq 0$. In order to get the set of orders of vertices, it suffices to observe that if e and e' are two edges exiting from v, such that $\mathrm{ord}_\varphi(e') = 2\,\mathrm{ord}_\varphi(e)$, then $\mathrm{ord}_\varphi(v) = \mathrm{ord}_\varphi(e)$, which gives the orders $m, 2m, \dots, 2^h m$ with $k-1 \leq h \leq k$. On the other hand, tripods of edges whose order has the form $2^j m/3$ may appear, like in point 3, for any $j = 0, \dots, k$. $\square$

By a *path* α between the (possibly coinciding) vertices v and w in a graph Γ we mean a (possibly non-simple) chain of edges having v and w as its ends. We call the number $\mathrm{len}(\alpha)$ of (non-necessarily distinct) edges in the chain the *length* of α. Moreover, we denote by $\bar{\alpha}$ the reversed path.

The next four lemmas concern minimal paths between vertices in a given φ-orbit.

Lemma 2.3. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a (possibly disconnected) uni/trivalent graph Γ. Given any (simple) path $\alpha \subset \Gamma$ of minimal length among all paths joining any two different vertices of a given φ-orbit, let v and v' be the ends of α in that φ-orbit. If $i \neq j$ and the four points $\varphi^i(v), \varphi^i(v'), \varphi^j(v)$ and $\varphi^j(v')$ are all distinct, then the paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ are disjoint.*

Proof. Let α be a path as in the statement. By contradiction, assume $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ are not disjoint for some $i \neq j$ satisfying the required condition. Then, they share at least a common internal edge e, and we have one of the two situations depicted in Figure 6, depending on the fact that e is traversed in the same direction

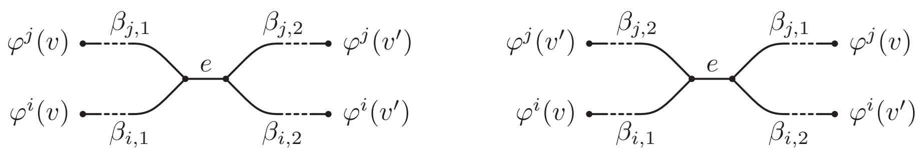

FIGURE 6. No common edge e between $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ with four distinct ends.

or not when going from $\varphi^i(v)$ to $\varphi^i(v')$ along $\varphi^i(\alpha)$ and from $\varphi^j(v)$ to $\varphi^j(v')$ along $\varphi^j(\alpha)$. Notice that the paths $\beta_{i,1}$ and $\beta_{i,2}$ forming $\varphi^i(\alpha) - \mathrm{Int}(e)$ are not necessarily disjoint from the paths $\beta_{j,1}$ and $\beta_{j,2}$ forming $\varphi^j(\alpha) - \mathrm{Int}(e)$. In any case, we find a path shorter than α between different vertices in the given φ-orbit. Namely, such path is either $\beta_{i,1}\overline{\beta}_{j,1}$ or $\beta_{i,2}\overline{\beta}_{j,2}$ in the left side case, while it is either $\beta_{i,1}\overline{\beta}_{j,2}$ or $\beta_{i,2}\overline{\beta}_{j,1}$ in the right side case. $\square$

Lemma 2.4. *Let $\varphi : \Gamma \to \Gamma$, $\alpha \subset \Gamma$, v and v', be as in Lemma 2.3. If $i \neq j$ with $\varphi^j(v) = \varphi^i(v')$ and $\varphi^j(v') \neq \varphi^i(v)$, then the union $\varphi^i(\alpha) \cup \varphi^j(\alpha)$ has the structure shown in Figure 7, where: 1) $k \geq 1$; 2) the paths $\delta_{i,1}, \dots, \delta_{i,k} \subset \varphi^i(\alpha)$ are disjoint from the paths $\delta_{j,1}, \dots, \delta_{j,k} \subset \varphi^j(\alpha)$; 3) $\mathrm{len}(\delta_{i,h}) = \mathrm{len}(\delta_{j,k-h+1}) \geq 1$ for every $h = 1, \dots, k$; 4) $\mathrm{len}(\delta_{i,1}) = \mathrm{len}(\delta_{j,k}) \geq \mathrm{len}(\alpha)/2$; 5) all the common paths shared by $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ have length ≥ 1, except the terminal one ending at $\varphi^j(v) = \varphi^i(v')$, which can have length zero.*

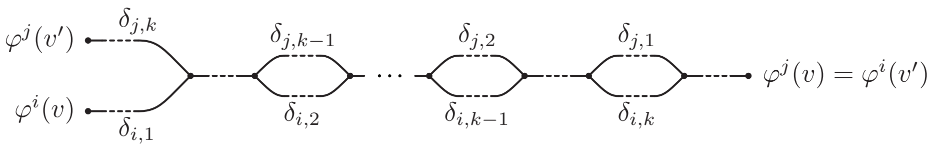

FIGURE 7. $\varphi^i(\alpha) \cup \varphi^j(\alpha)$ for $\varphi^j(v) = \varphi^i(v')$ and $\varphi^j(v') \neq \varphi^i(v)$.

Proof. Look at Figure 6 assuming $\varphi^j(v) = \varphi^i(v')$ and $\varphi^j(v') \neq \varphi^i(v)$. By a similar argument as in the proof of the previous lemma, the minimality of α implies that $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ cannot share an edge e like in the left side of the figure, while if $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ share an edge e like in the right side of the figure then $\mathrm{len}(\beta_{i,1}) = \mathrm{len}(\beta_{j,2}) \geq \mathrm{len}(\alpha)/2$. This gives properties 1 and 4.

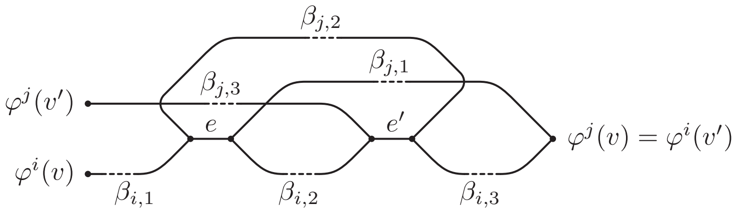

FIGURE 8. Common edges e and e' cannot occur in the same order along the paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ in Lemma 2.4.

For property 2, it suffices to observe that, if e and e' are two common edges of $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$, then they occur in opposite orders along the two paths starting from $\varphi^i(v)$ and $\varphi^i(v')$ respectively. Indeed, in the contrary case, we would have $\varphi^i(\alpha) = \beta_{i,1} e \beta_{i,2} e' \beta_{i,3}$ and $\varphi^j(\alpha) = \beta_{j,1} \bar{e} \beta_{j,2} \bar{e}' \beta_{j,3}$ like in Figure 8, where β_i's and the β_j's are possibly empty and non-necessarily disjoint. Then, the minimality of $\varphi^i(\alpha)$ would imply $\mathrm{len}(\beta_{i,2}) + 2 \leq \mathrm{len}(\beta_{j,2})$, while the minimality of $\varphi^j(\alpha)$ would imply $\mathrm{len}(\beta_{j,2}) + 2 \leq \mathrm{len}(\beta_{i,2})$, which would be absurd.

At this point, property 3 and 5 immediately follow from the minimality of $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ and from the trivalency of the vertices respectively. $\square$

Lemma 2.5. *Let $\varphi : \Gamma \to \Gamma$, $\alpha \subset \Gamma$, v and v', be as in Lemma 2.3. If $i \neq j$ with $\varphi^j(v) = \varphi^i(v')$ and $\varphi^j(v') = \varphi^i(v)$, then the union $\varphi^i(\alpha) \cup \varphi^j(\alpha)$ has the structure shown in Figure 9, where: 1) $k \geq 0$; 2) the paths $\delta_{i,1}, \ldots, \delta_{i,k} \subset \varphi^i(\alpha)$ are disjoint from the paths $\delta_{j,1}, \ldots, \delta_{j,k} \subset \varphi^j(\alpha)$; 3) $\mathrm{len}(\delta_{i,h}) = \mathrm{len}(\delta_{j,k-h+1}) \geq 1$ for every $h = 1, \ldots, k$; 4) all the common paths shared by $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ have length ≥ 1, except both the terminal ones that can have length zero.*

Moreover, this may happen only when $\mathrm{ord}(\varphi)$ is even.

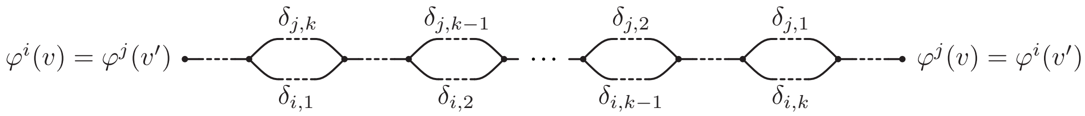

FIGURE 9. $\varphi^i(\alpha) \cup \varphi^j(\alpha)$ for $\varphi^i(v) = \varphi^j(v')$ and $\varphi^j(v) = \varphi^i(v')$.

Proof. The same arguments in the proof of Lemma 2.4 still work here, to give the first part of the statement. In fact, in that proof the assumption that $\varphi^j(v') \neq \varphi^i(v)$ was only used to obtain $k \geq 1$ and property 4 (in the statement of that lemma).

For the last sentence, we observe that the conditions $\varphi^j(v) = \varphi^i(v')$ and $\varphi^j(v') = \varphi^i(v)$ imply that φ^{j-i} swaps v and v', which in turn implies that $\mathrm{ord}(\varphi^{j-i})$, and hence $\mathrm{ord}(\varphi)$ as well, is even. $\square$

Lemma 2.6. *Let $\varphi : \Gamma \to \Gamma$, $\alpha \subset \Gamma$, v and v', be as in Lemma 2.3. If $i \neq j$ with $\varphi^i(v) = \varphi^j(v)$ and $\varphi^i(v') = \varphi^j(v')$, then the union $\varphi^i(\alpha) \cup \varphi^j(\alpha)$ has the structure shown in Figure 10, where: 1) $k \geq 0$; 2) the paths $\delta_{i,1}, \dots, \delta_{i,k} \subset \varphi^i(\alpha)$ are disjoint from the paths $\delta_{j,1}, \dots, \delta_{j,k} \subset \varphi^j(\alpha)$; 3) $\mathrm{len}(\delta_{i,h}) = \mathrm{len}(\delta_{j,h}) \geq 1$ for every $h = 1, \dots, k$; 4) all the common paths shared by $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ have length ≥ 1, except both the terminal ones that can have length zero.*

Moreover, apart for the trivial case of $\varphi^i(\alpha) = \varphi^j(\alpha)$, this may happen only when $\mathrm{ord}(\varphi)$ is either even or an odd multiple of 3, being in the latter case $\varphi^i(\alpha) = \delta_{i,1}$ and $\varphi^j(\alpha) = \delta_{j,1}$.

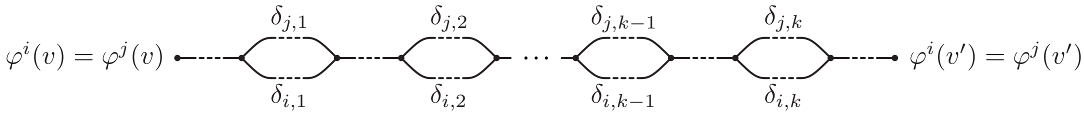

FIGURE 10. $\varphi^i(\alpha) \cup \varphi^j(\alpha)$ for $\varphi^i(v) = \varphi^j(v)$ and $\varphi^i(v') = \varphi^j(v')$.

Proof. The proof of the first part of the statement is completely analogous to that of the previous lemma, except for the orderings of the δ_i's along $\varphi^i(\alpha)$ and of the δ_j's along $\varphi^j(\alpha)$, which now coincide.

To prove the second part, assume that $\varphi^i(\alpha) \neq \varphi^j(\alpha)$ and $\mathrm{ord}(\varphi)$ is odd. Then, we have $\varphi^i(\alpha) = \delta_{i,1}$ and $\varphi^j(\alpha) = \delta_{j,1}$. Otherwise, φ^{j-i} would fix some edge of $\varphi^i(\alpha) \cap \varphi^j(\alpha)$ and $\mathrm{ord}(\varphi^{j-i})$ should be even by Lemma 2.2, in contrast with the oddness of $\mathrm{ord}(\varphi)$. For the same reason φ^{j-i} cannot swap $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$, then it cyclically permutes the three paths $\varphi^i(\alpha)$, $\varphi^j(\alpha)$ and $\varphi^{2j-i}(\alpha)$, which meet each other only at their common ends $\varphi^i(v)$ and $\varphi^i(v')$. This implies that $\mathrm{ord}(\varphi^{j-i})$, and hence $\mathrm{ord}(\varphi)$ as well, is a multiple of 3. $\square$

3. The case of order p^m with p prime greater than 3

The aim of this section is to show that any automorphism $\varphi \in \mathcal{A}_{g,b}$ with order $\mathrm{ord}(\varphi) = p^m$ for a prime $p > 3$ is F-equivalent to the composition of two automorphisms of order 2.

Actually, we will prove this fact under the weaker assumption that $n = \mathrm{ord}(\varphi)$ is a multiple of neither 2 nor 3. In fact, all the arguments are only based on the following special properties, which hold for any automorphism $\varphi : \Gamma \to \Gamma$ of a (possibly disconnected) uni/trivalent graph Γ, having such an order n:

1) $\mathrm{ord}_\varphi(v) = \mathrm{ord}_\varphi(e) = n$ for any vertex v and edge e of Γ, thanks to Lemma 2.2;
2) the situations of Lemmas 2.5 and 2.6 cannot occur.

Lemma 3.1. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a (possibly disconnected) uni/trivalent graph Γ, whose order $n = \mathrm{ord}(\varphi)$ is a multiple of neither 2 nor 3. If $\alpha \subset \Gamma$ is a (simple) path of minimal length among all the paths joining any two distinct φ-equivalent vertices, then $\cup_i \varphi^i(\alpha)$ is a disjoint union $C \sqcup \varphi(C) \sqcup$*

$\ldots \sqcup \varphi^{n/\ell-1}(C)$ of n/ℓ simple cycles (possibly a single one, for $\ell = n$), each given by a concatenation of ℓ images of α with ℓ a divisor of n greater than 1. In fact, there exists a positive integer $s = t\,n/\ell < n$ with $(t,\ell) = 1$, such that the cycle C is given by $\alpha\,\varphi^s(\alpha)\cdots\varphi^{(\ell-1)s}(\alpha)$. Moreover, up to F-equivalence we can assume α to consist of a single edge a (see Figure 11, where a_i stands for $\varphi^{is}(a)$ and a similar notation is adopted for the v_i's as well).

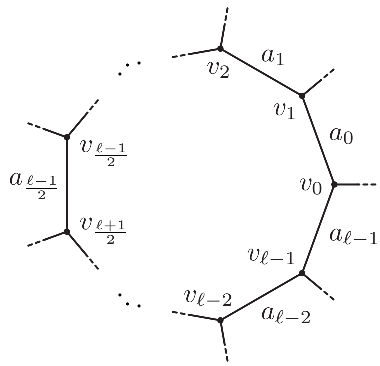

FIGURE 11. The form of the cycle C in Lemma 3.1.

Proof. Let α be a path as in the statement and let $v \neq v'$ be its ends. For any $i \neq j \bmod n$, let us consider the two paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$, and the four different situations described in Lemmas 2.3 to 2.6, which cover all the possibilities, thanks to the φ-equivalence of v and v'.

The situations of Lemmas 2.5 and 2.6 cannot occur, hence in particular $\varphi^i(\alpha) \neq \varphi^j(\alpha)$. Therefore, $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ can have non-empty intersection only as in Lemma 2.4. In this situation, if $\varphi^i(u)$ with $u \in \alpha$ is the common end of $\delta_{i,1}$ and $\delta_{j,k}$, then $\varphi^j(u) \in \delta_{j,k}$ since $\operatorname{len}(\delta_{i,1}) = \operatorname{len}(\delta_{j,k}) \geq \operatorname{len}(\alpha)/2$. Actually, $\varphi^j(u)$ has to coincide with either $\varphi^j(v')$ or $\varphi^i(u)$, otherwise the global minimality of α would be contradicted.

Since $\varphi^j(u) = \varphi^i(u)$ is impossible, being $\operatorname{ord}_\varphi(u) = n$, we are left with the unique possibility $\varphi^j(u) = \varphi^j(v')$, that is $u = v'$. Then, we have $\varphi^i(\alpha) = \delta_{i,1}$ and $\varphi^j(\alpha) = \delta_{j,k}$, which meet only at their common end $\varphi^j(v) = \varphi^i(v')$. Moreover, such end cannot be shared by any other $\varphi^h(\alpha)$. Otherwise, either $\varphi^i(\alpha)$ and $\varphi^h(\alpha)$ or $\varphi^j(\alpha)$ and $\varphi^h(\alpha)$ would be in the situation of Lemma 2.6.

Then, we can conclude in a straightforward way that $\cup_i\,\varphi^i(\alpha)$ has the stated form, with $s < n$ uniquely determined by $v' = \varphi^s(v)$, ℓ the smallest positive integer such that $\varphi^{\ell s}(v) = v$, and $t = \ell\,s/n$ (this is an integer since $\operatorname{ord}_\varphi(v) = n$ and it is coprime with ℓ by the minimality of ℓ).

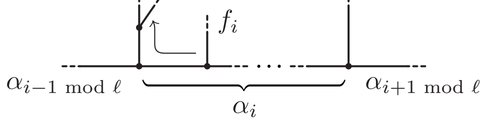

FIGURE 12. Reducing $\operatorname{len}(\alpha)$ in Lemma 3.1.

At this point, we are left to prove that the path α can be replaced by a single edge up to F-equivalence. Proceeding by induction, it suffices to show how to reduce len(α) whenever this is greater than 1. Such reduction can be achieved by sliding all the copies $f_i = \varphi^{is}(f)$ of the unique edge f attached at the first intermediate vertex of α but not belonging to α, so that their ends are slided out of the paths $\alpha_i = \varphi^{is}(\alpha)$, as indicated in Figure 12. According to Remark 1.5, those slidings correspond to a φ-invariant F-move, given by simultaneous elementary F-moves performed on the first edges of all the α_i's, which are pairwise disjoint if $\mathrm{len}(\alpha) \geq 2$. $\square$

Lemma 3.2. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a (possibly disconnected) uni/trivalent graph Γ, whose order $n = \mathrm{ord}(\varphi)$ is a multiple of neither 2 nor 3. If $\alpha \subset \Gamma$ is a (simple) path of minimal length among all paths joining any two distinct (hence disjoint, see proof) terminal edges in a given φ-orbit, then up to F-equivalence we can assume $\cup_i \varphi^i(\alpha)$ to be a disjoint union $C \sqcup \varphi(C) \sqcup \ldots \sqcup \varphi^{n/\ell-1}(C)$ of n/ℓ cycles (possibly a single one, for $\ell = n$), with ℓ a divisor of n greater than 1 and $C = \alpha\,\varphi^s(\alpha) \cdots \varphi^{(\ell-1)s}(\alpha)$ for some positive integer $s = t\,n/\ell < n$ with $(t, \ell) = 1$, like in Lemma 3.1. Moreover, α can be assumed to consist of either one edge a or two edges a and b (see Figure 13, where a_i and b_i stand for $\varphi^{is}(a)$ and $\varphi^{is}(b)$ respectively, and a similar notation is adopted for the terminal edges e_i's and the vertices v_i's as well).*

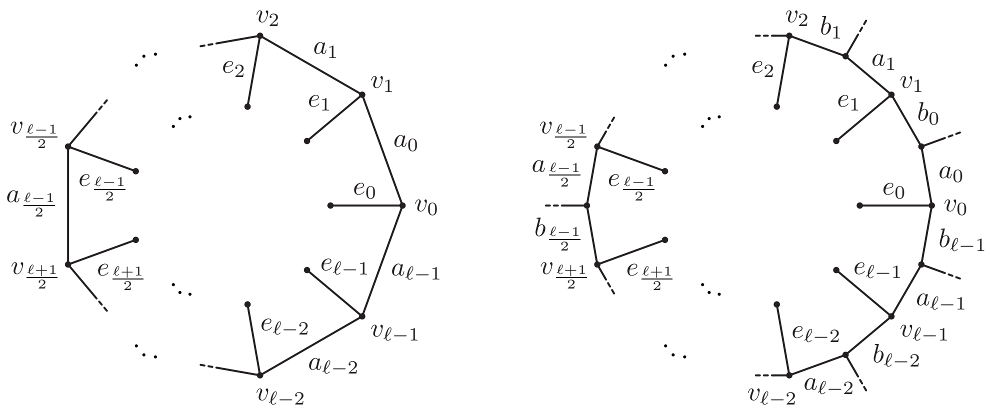

FIGURE 13. The two possible forms of the cycle C in Lemma 3.2.

Proof. Let α be a path as in the statement, and $e \neq e'$ be the terminal edges it joins. Then the ends of α coincide with the unique trivalent ends v and v' of e and e' respectively. Notice that $v \neq v'$, being $\mathrm{ord}_\varphi(v) = \mathrm{ord}_\varphi(e) = n$ by Lemma 2.2.

For any $i \neq j \bmod n$, by arguing as in the proof of Lemma 3.1, we can prove that $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ can meet only as in Lemma 2.4, and that their common end $\varphi^j(v) = \varphi^i(v')$ cannot be shared by any other $\varphi^h(\alpha)$. The same Lemma 2.4 also tells us that $\mathrm{len}(\delta_{i,1}) = \mathrm{len}(\delta_{j,k}) \geq \mathrm{len}(\alpha)/2$ (cf. Figure 7). But here the equality cannot occur, otherwise φ^{j-i} would fix the common end of $\delta_{i,1}$ and $\delta_{j,k}$, and it would cyclically permute the three edges exiting from it, in contrast with the assumption that n is not a multiple of 3.

Then, we can conclude straightforwardly that $\cup_i \varphi^i(\alpha)$ consists of the disjoint union $C \sqcup \varphi(C) \sqcup \ldots \sqcup \varphi^{n/\ell-1}(C)$, with $s < n$ uniquely determined by $\varphi^s(v) = v'$, ℓ the smallest positive integer such that $\varphi^{\ell s}(v) = v$, $t = s\,\ell/n$ (cf. proof of Lemma 3.1), and $C = \alpha\,\varphi^s(\alpha) \cdots \varphi^{(\ell-1)s}(\alpha)$ having the form depicted in Figure 14. Here, and in the next figure as well, α_i stands for the path $\varphi^{is}(\alpha)$, which is the part of C in the corresponding circular sector, and each arc represents a path of edges.

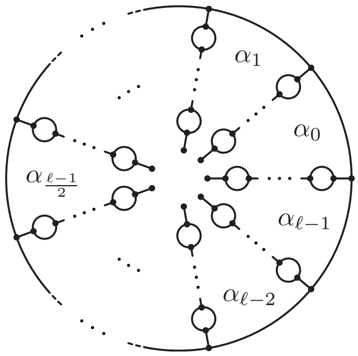

FIGURE 14. The starting configuration C in the proof of Lemma 3.2.

In the very special case where the paths α_i consist of single edges, they can only share their common ends. Hence, the configuration C in Figure 14 coincides with that in the left side of Figure 13, and we are done.

Otherwise, by a sequence of φ-invariant slidings (cf. Remark 1.5) on the edges attached to C and to its copies, as indicated in Figure 15 for three of such edges, we can reduce all the arcs in Figure 14 to single edges, possibly except the ones forming the big circle. In the same way, when the first edge of α_i coincides with the last edge of $\alpha_{i-1 \bmod \ell}$, we also fuse it with the edge $e_i = \varphi^{is}(e)$.

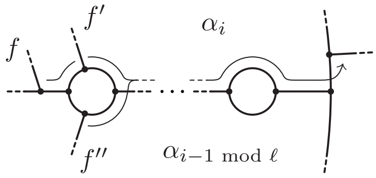

FIGURE 15. Simplifying the starting configuration.

Then, we can eliminate all the resulting bigons, by a φ-invariant F-move, which acts on C by simultaneous elementary F-moves on the edges between any two consecutive of them and between the most external ones and the big circle, as shown in Figure 16.

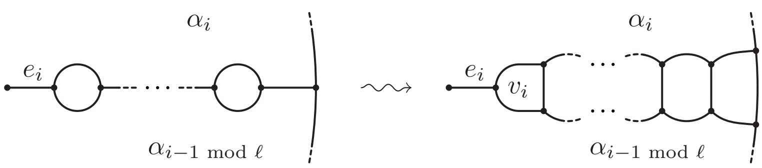

FIGURE 16. Eliminating the bigons by simultaneous edge F-moves.

Finally, to get the configuration in the right side of Figure 13, it remains to reduce α to a chain of only two edges a and b. This can be done proceeding by induction, like in the last part of the proof of Lemma 3.1. In Figure 17 we see how to reduce $\operatorname{len}(\alpha)$ whenever this is greater than 2. Here, $a_i = \varphi^{is}(a)$ and $f_i = \varphi^{is}(f)$ are respectively the images in $\alpha_i = \varphi^{is}(\alpha)$ of the first edge a of α and of the edge f attached at the second intermediate vertex of α but not belonging to α. The desired reduction is achieved through an invariant F-move, which simultaneously slides all

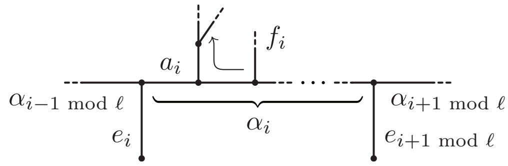

FIGURE 17. Reducing $\mathrm{len}(\alpha)$ in Lemma 3.2.

the f_i, in such a way that their ends pass from the α_i's to the first edges attached to them. $\square$

Notice that, in Lemma 3.2 we cannot further reduce α to a single edge by a slide letting even the last intermediate point of α pass to e, as we did in Lemma 3.1.

Proposition 3.3. *Let $\varphi \in \mathcal{A}_{g,b}$ be a non-trivial automorphism, whose order $n = \mathrm{ord}(\varphi) > 1$ is a multiple of neither 2 nor 3. Then, φ is F-equivalent to the composition $\tau \circ \sigma$ of two automorphisms $\sigma, \tau \in \mathcal{A}_{g,b}$ such that $\mathrm{ord}(\sigma) = \mathrm{ord}(\tau) = 2$. Moreover, both σ and τ can be assumed to fix an edge and reverse an invariant edge.*

Proof. Let $\varphi : \Gamma \to \Gamma$ any automorphism in $\mathcal{A}_{g,b}$ as in the statement. We will construct by recursion a sequence $\varphi_i : \Gamma_i \to \Gamma_i$ of automorphisms in $\mathcal{A}_{g,b}$ and a sequence of subgraphs $\Lambda_i \subset \Gamma_i$ with $i = 0, \ldots, r$ and $r \geq 2$, having the following properties:

1) $\varphi_0 = \varphi$ (hence $\Gamma_0 = \Gamma$), and φ_i is F-equivalent to φ_{i-1} for every $1 \leq i \leq r$;
2) $\emptyset = \Lambda_0 \subsetneq \Lambda_1 \subsetneq \ldots \subsetneq \Lambda_r = \Gamma_r$, and the subgraph $\Delta_i = \mathrm{Cl}(\Gamma_i - \Lambda_i)$, given by the closure in Γ_i of the difference $\Gamma_i - \Lambda_i$, is a uni/trivalent graph whose intersection with Λ_i is a single φ_i-orbit $U_i \subset \Delta_i$ of its free ends for every $1 \leq i \leq r-1$;
3) Λ_i is φ_i-invariant and $\varphi_{i|\Lambda_{i-1}} = \varphi_{i-1|\Lambda_{i-1}}$ for every $1 \leq i \leq r$;
4) $\varphi_{i|\Lambda_i} : \Lambda_i \to \Lambda_i$ is the composition $\tau_i \circ \sigma_i$ of two automorphisms $\sigma_i : \Lambda_i \to \Lambda_i$ and $\tau_i : \Lambda_i \to \Lambda_i$ such that $\mathrm{ord}(\sigma_i) = \mathrm{ord}(\tau_i) = 2$, for every $1 \leq i \leq r$; moreover, σ_i fixes exactly one vertex u_i in U_i (hence τ_i fixes $\varphi_i^{(n+1)/2}(u_i) \in U_i$) for every $1 \leq i \leq r-1$, and both σ_i and τ_i fix an edge and reverse an invariant edge for every $2 \leq i \leq r$.

This will prove the proposition, since $\sigma = \sigma_r$ and $\tau = \tau_r$ satisfy all the required conditions, being $\tau \circ \sigma = \tau_r \circ \sigma_r = \varphi_{r|\Lambda_r} = \varphi_r$ F-equivalent to φ in $\mathcal{A}_{g,b}$.

To start the construction, we apply Lemma 3.1 to the automorphism φ and a minimal path $\alpha \subset \Gamma$ as in that statement (this exists since Γ is connected but not reduced to a single vertex and φ is non-trivial). As a result, we get an automorphism $\varphi_1 : \Gamma_1 \to \Gamma_1$ in $\mathcal{A}_{g,b}$, which is F-equivalent to $\varphi_0 = \varphi$ and such that the minimal path corresponding to α in Γ_1 consists of only one (internal) edge $a \subset \Gamma_1$. Then, we put $\Lambda_1 = \cup_j \varphi_1^j(a) \subset \Gamma_1$, which is φ_1-invariant by definition. Lemma 3.1 tells us that there exist a divisor $\ell > 1$ of n and an integer $0 < t < l$ with $(t, \ell) = 1$, such that Λ_1 consists of the disjoint union $C \sqcup \varphi_1(C) \sqcup \ldots \sqcup \varphi_1^{n/\ell-1}(C)$ of n/ℓ simple cycles (possibly a single one, for $\ell = n$), where $C = a\, \varphi_1^s(a) \cdots \varphi_1^{(\ell-1)s}(a)$ with $s = t\, n/\ell$.

Let v be the first end of the edge a, so that $\varphi_1^s(v)$ is the second one. Then, the set of vertices of Λ_1 is the φ_1-orbit $V = \{v, \varphi_1(v), \ldots, \varphi_1^{n-1}(v)\}$, and any automorphism on Λ_1 is uniquely determined by its action on V, since n is odd and hence $\ell > 2$. Therefore, we can first define the two involutions σ_1 and τ_1 on V in such a way that $\tau_1 \circ \sigma_1$ coincides with the restriction of φ_1 to V, then check that they preserve adjacency of vertices in Λ_1 and hence give well-defined automorphisms of Λ_1.

We define σ_1 and τ_1 on V by putting $\sigma_1(\varphi_1^j(v)) = \varphi_1^{-j}(v)$ and $\tau_1(\varphi_1^j(v)) = \varphi_1^{1-j}(v)$, for $j = 0, \dots, n-1$. Clearly, these are involutions and their composition $\tau_1 \circ \sigma_1$ acts on V as φ_1. Concerning the preservation of adjacency, it suffices to observe that two elements $\varphi_1^j(v)$ and $\varphi_1^k(v)$ of V are adjacent vertices in Λ_1 if and only if $|j-k| = s \bmod n$, and both σ_1 and τ_1 preserve this condition.

Finally, we note that $\Delta_1 = \mathrm{Cl}(\Gamma_1 - \Lambda_1)$ is a uni/trivalent graph and $\Delta_1 \cap \Lambda_1$ is given by the φ_1-orbit $U_1 = V$ of free ends of Δ_1, as required in point 2 above. Moreover, σ_1 fixes only the vertex $u_1 = v$ of U_1 and reverses the edge $\varphi_1^{(n+1)/2}(a)$, while τ_1 reverses the edge a, as required in point 4 above.

Now, to realize the recursive step of the construction, assume we are given $\varphi_{i-1} : \Gamma_{i-1} \to \Gamma_{i-1}$, $u_{i-1} \in U_{i-1} \subset \Lambda_{i-1} \subsetneq \Gamma_{i-1}$ and $\sigma_{i-1}, \tau_{i-1} : \Lambda_{i-1} \to \Lambda_{i-1}$ satisfying all the properties stated in the above points 1 to 4 for $i-1 < r$. Lemmas 2.1 and 2.2 ensure that $\mathrm{ord}(\varphi_{i-1}) = n$ and that the cardinality of the φ_{i-1}-orbit U_{i-1} is equal to n as well. Hence, also the restriction of φ_{i-1} to the uni/trivalent graph $\Delta_{i-1} = \mathrm{Cl}(\Gamma_{i-1} - \Lambda_{i-1})$ is an automorphism of the same order n. Furthermore, the terminal edges of Δ_{i-1} ending at vertices in U_{i-1} are all distinct. In fact, if two of them would coincide, then their φ-order would be even, which contradicts the hypothesis that n is odd.

If Δ_{i-1} does not contain any path joining different terminal edges ending at vertices in U_{i-1}, then it consists of n components each containing a single vertex in U_{i-1}. Denoting by C the component of Δ_{i-1} containing that vertex u_{i-1}, we have the component decomposition $\Delta_{i-1} = C \sqcup \varphi_{i-1}(C) \sqcup \ldots \sqcup \varphi_{i-1}^{n-1}(C)$, with φ_{i-1} cyclically permuting the components. In this case, we put $\varphi_i = \varphi_{i-1}$ and $\Lambda_i = \Gamma_i = \Gamma_{i-1}$. Moreover, we define σ_i and τ_i as the unique automorphisms of Γ_i extending σ_{i-1} and τ_{i-1}, in such a way that they act on each $\varphi_{i-1}^j(C)$ with $j = 0, \dots, n-1$, as φ_{i-1}^{-2j} and φ_{i-1}^{1-2j} respectively. A straightforward verification shows that such φ_i, Λ_i, σ_i and τ_i satisfy all the properties stated in the above points 1 to 4 for the case when $i = r$, which means that this step terminates the recursion.

Otherwise, if a path $\alpha \subset \Delta_{i-1}$ exists joining different terminal edges of Δ_{i-1} ending at vertices in U_{i-1}, then we can choose it to be minimal (hence simple) and apply Lemma 3.2 to the restriction of φ_{i-1} to Δ_{i-1} and to such a minimal α, in order to get the structure described in that statement for $\cup_j \varphi_{i-1}^j(\alpha) \subset \Delta_{i-1}$ up to F-equivalence. Such an F-equivalence only involves internal edges of Δ_{i-1}, hence it does not change the set of free ends $U_{i-1} \subset \Delta_{i-1}$ and the restriction of $\varphi_{i-1|\Delta_{i-1}}$. Therefore, it can be extended to an F-equivalence of the whole φ_{i-1} on Γ_{i-1}, which leaves Λ_{i-1} and the restriction $\varphi_{i-1|\Lambda_{i-1}}$ unchanged.

As a result we get a new uni/trivalent graph Γ_i, such that $\Lambda_{i-1} \subset \Gamma_i$ and a new automorphism $\varphi_i : \Gamma_i \to \Gamma_i$, which is F-equivalent to φ_{i-1} and coincides with φ_{i-1} on Λ_{i-1}. We also get a new minimal path $\alpha \subset \mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$, which joins different φ_i-equivalent terminal edges of $\mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$ ending at vertices in U_{i-1}, such that $\cup_j \varphi_i^j(\alpha)$ itself (no more up to F-equivalence) has the structure described in Lemma 3.2. Let $e \subset \Gamma_i$ be the terminal edge of $\mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$ ending at u_{i-1}, and $v \in \Gamma_i$ be the other end of e, which is a trivalent vertex of $\mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$. Up to φ_i-equivalence, we can assume that $\alpha \subset \Gamma_i$ starts at v and ends at $\varphi_i^s(v)$ with $0 < s < n$.

According to Lemma 3.2, we can also assume that α consists of either one edge a or two edges a and b sharing the vertex u, while $\cup_j \varphi_i^j(\alpha)$ consists of the disjoint union $C \sqcup \varphi_i(C) \sqcup \ldots \sqcup \varphi_i^{n/\ell-1}(C)$ of n/ℓ cycles (possibly a single one, for $\ell = n$), where $C = \alpha\, \varphi_i^s(\alpha) \cdots \varphi_i^{(\ell-1)s}(\alpha)$ for some $\ell > 1$ such that $s\ell = tn$ with $(t, \ell) = 1$.

We put $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_i^j(e) \cup_j \varphi_i^j(\alpha)$ and define σ_i and τ_i to be the unique automorphisms of Λ_i extending σ_{i-1} and τ_{i-1} respectively. To see that such extensions exist, we first define $\sigma_i(\varphi_i^j)(v) = \varphi_i^{-j}(v)$ and $\tau_i(\varphi_i^j)(v) = \varphi_i^{1-j}(v)$, and also $\sigma_i(\varphi_i^j)(u) = \varphi_i^{-j}(u)$ and $\tau_i(\varphi_i^j)(u) = \varphi_i^{1-j}(u)$ in the case when $\alpha = ab$, for every $j = 0, \ldots, n-1$. Then, essentially by the same argument used above for $i = 1$, we check that these definitions, together with $\sigma_i(\varphi_i^j)(u_{i-1}) = \varphi_i^{-j}(u_{i-1})$ and $\tau_i(\varphi_i^j)(u_{i-1}) = \varphi_i^{1-j}(u_{i-1})$ for every $i = 0, \ldots, n$, preserve adjacency of vertices in Λ_i, hence they determine automorphisms σ_i and τ_i of the graph Λ_i extending σ_{i-1} and τ_{i-1} respectively. Notice that $\tau_i \circ \sigma_i$ trivially coincides with $\varphi_{i|\Lambda_i}$. Moreover, σ_i fixes the edge e and reverses the edge $\varphi_i^{(n+1)/2}(a)$, while τ_i fixes the edge $\varphi_i^{(n+1)/2}(e)$ and reverses the edge a. Finally, $u_i = \varphi_i^{(n+1)/2}(u)$ is the unique vertex in its φ_i-orbit $U_i = \{\varphi_i^j(u) \mid j = 0, \ldots, n\}$ fixed by σ_i.

At this point, if α consists of the only edge a, hence C has the structure depicted in the left side of Figure 13, then $\Lambda_i = \Gamma_i$ by the connectedness of Γ_i and the recursion terminates with $r = i$. Otherwise, if α consists of the two edges a and b, hence C has the structure depicted in the right side of Figure 13, then we put $\Delta_i = \mathrm{Cl}(\Gamma_i - \Lambda_i)$ and observe that this is a uni/trivalent graph such that $\Delta_i \cap \Lambda_i = U_i$. This conclude the recursive step. $\square$

4. The case of order 3^m

In this section, we want to prove that any automorphism $\varphi \in \mathcal{A}_{g,b}$ with order $\mathrm{ord}(\varphi) = 3^m$ is F-equivalent to the composition of two automorphisms of order 2.

Like in the previous section, a weaker assumption will suffice to get the same conclusion. Namely, we will assume that $n = \mathrm{ord}(\varphi)$ is an odd multiple of 3. The only properties we will need, for any automorphism $\varphi : \Gamma \to \Gamma$ of a (possibly disconnected) uni/trivalent graph Γ with such an order n, are the following:

1) $\mathrm{ord}_\varphi(e) = n$ for any edge e of Γ, while $\mathrm{ord}_\varphi(v)$ is either n or $n/3$ for any vertex v of Γ, thanks to Lemma 2.2;
2) the situation of Lemmas 2.5 cannot occur, while that of Lemma 2.6 can only occur with the two paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ sharing no edge.

The whole argument is essentially the same as in the previous section, except for some more cases occurring here. Hence, in both statements and proofs we will just concern with those extra cases, while referring for the others to the analogous statements and proofs in the previous section.

Lemma 4.1. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a (possibly disconnected) uni/trivalent graph Γ, whose order $n = \mathrm{ord}(\varphi)$ is an odd multiple of 3. If $\alpha \subset \Gamma$ is a (simple) path of minimal length among all paths joining any two distinct φ-equivalent vertices, then $\cup_i \varphi^i(\alpha)$ is one of the following:*

1) *a disjoint union $C \sqcup \varphi(C) \sqcup \ldots \sqcup \varphi^{n/\ell-1}(C)$ of n/ℓ simple cycles (possibly a single one, for $\ell = n$), each given by a concatenation of images of α, having the same*

structure as described in Lemma 3.1; in this case, up to F-equivalence we can assume α to consist of a single edge a (see Figure 11, where a_i stands for $\varphi^{is}(a)$ and a similar notation is adopted for the vertices v_i as well);

2) *a disjoint union $T \sqcup \varphi(T) \sqcup \ldots \sqcup \varphi^{n/3-1}(T)$ of $n/3$ tripods (possibly a single one, for $n = 3$), where $T = \alpha \cup \varphi^{n/3}(\alpha) \cup \varphi^{2n/3}(\alpha)$ and α is the concatenation of two edges a and b, such that $b = \varphi^{n/3}(\bar{a})$, $\varphi^{n/3}(b) = \varphi^{2n/3}(\bar{a})$ and $\varphi^{2n/3}(b) = \bar{a}$ (see Figure 18, where a_i stands for $\varphi^{in/3}(a)$ and a similar notation is adopted for the vertices v_i, which can be either all univalent as on the left side or all trivalent as on the right side).*

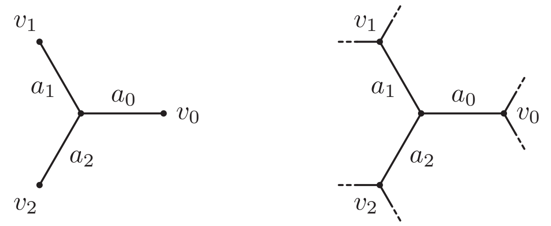

Figure 18. The form of the tripod T in Lemma 4.1.

Proof. Let $\alpha \subset \Gamma$ be an arbitrary path of minimal length among all paths joining any two distinct φ-equivalent vertices and let $v \neq v'$ be its ends. For any $i \neq j \bmod n$, let us consider the two paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$, and the four different situations described in Lemmas 2.3 to 2.6, which cover all the possibilities, thanks to the φ-equivalence of v and v'.

The situation of Lemma 2.5 cannot occur, while that of Lemma 2.6 could possibly occur only with $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ sharing no edge, but the argument below shows this is actually impossible.

Assume by contradiction that $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ are as in Lemma 2.6, hence they meet at $\varphi^i(v) = \varphi^j(v)$ and $\varphi^i(v') = \varphi^j(v')$. Let us denote by a the first edge of α starting from v and by u the other end of a. Then, the vertices $\varphi^i(u)$ and $\varphi^j(u)$ are distinct since $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ do not share any edge. Being n an odd multiple of 3, thanks to Lemma 2.2 we have $\mathrm{ord}_\varphi(v) = n/3$ and $\mathrm{ord}_\varphi(u) = n$.

As a consequence, we get $\mathrm{len}(\alpha) = 2$. In fact, the concatenation $\varphi^i(\bar{a})\,\varphi^j(a)$ is simple path of length 2 between the two distinct φ-equivalent vertices $\varphi^i(u)$ and $\varphi^j(u)$. Hence, by the global minimality of α we have $\mathrm{len}(\alpha) \leq 2$. On the other hand, α cannot be reduced to the single edge a, otherwise v and u should be φ-equivalent, in contrast with $\mathrm{ord}_\varphi(u) \neq \mathrm{ord}_\varphi(v)$.

Now, let s be an integer such that $v' = \varphi^s(v)$. Then, $\varphi^s(u)$ coincides with one of u, $\varphi^{n/3}(u)$ and $\varphi^{2n/3}(u)$, since these are the three vertices adjacent to v'. In any case, we can conclude that s is a multiple of $n/3$, which is in contrast with $\mathrm{ord}_\varphi(v) = n/3$ and $v' \neq v$. This proves that the situation of Lemma 2.6 cannot occur.

At this point, we have that $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ can only meet as in Lemma 2.4. Let $\varphi^i(u)$ with $u \in \alpha$ be the common end of $\delta_{i,1}$ and $\delta_{j,k}$. As in the proof of Lemma 3.1, we can see that either $\varphi^j(u) = \varphi^j(v')$ or $\varphi^j(u) = \varphi^i(u)$, due to the global minimality of α. But differently from that proof, here the latter equality can actually hold when u has order $n/3$ and $j = i \bmod n/3$.

If $\varphi^j(u) = \varphi^i(u)$, then the global minimality of α implies that both $\delta_{i,1}$ and $\delta_{j,k}$ have length 1. Therefore, α is the concatenation of two edges a and b sharing the vertex u, and hence $T = \alpha \cup \varphi^{n/3}(\alpha) \cup \varphi^{2n/3}(\alpha)$ has the form described in point 2 of the statement. Of course, this means that $\cup_i \varphi^i(\alpha) = T \cup \varphi(T) \cup \ldots \cup \varphi^{n/3-1}(T)$. Moreover, any two different copies of T in this union are disjoint, otherwise they should share a free end, and there would be two paths φ-equivalent to α exiting from that common vertex like in Lemma 2.6.

Notice that the last argument shows that, if $\varphi^j(u) = \varphi^i(u)$ for some i and j, then the same holds for any i and j such that $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ are not disjoint.

So, we are left with the case when $\varphi^j(u) = \varphi^j(v')$ for any two non-disjoint paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$. In this case, the same argument exploited in the proof of Lemma 3.1, allows us to conclude that the situation is the one described in point 1 of the statement. $\square$

Lemma 4.2. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a (possibly disconnected) uni/trivalent graph Γ, whose order $n = \mathrm{ord}(\varphi)$ is an odd multiple of 3. If $\alpha \subset \Gamma$ is a (simple) path, possibly degenerated to a single vertex, of minimal length among all paths joining any two distinct terminal edges in a given φ-orbit, then up to F-equivalence we can assume $\cup_i \varphi^i(\alpha)$ to be one of the following:*

1) *a disjoint union $C \sqcup \varphi(C) \sqcup \ldots \sqcup \varphi^{n/\ell-1}(C)$ of n/ℓ simple cycles (possibly a single one, for $\ell = n$), each given by a concatenation of images of α, having the same structure as described in Lemma 3.2, with α consisting of either one edge a or two edges a and b (see Figure 13, where a_i and b_i stand for $\varphi^{is}(a)$ and $\varphi^{is}(b)$ respectively, and a similar notation is adopted for the terminal edges e_i's and the vertices v_i's as well);*
2) *a disjoint union $T \sqcup \varphi(T) \sqcup \ldots \sqcup \varphi^{n/3-1}(T)$ of $n/3$ tripods (possibly a single one, for $n = 3$), where $T = \alpha \cup \varphi^{n/3}(\alpha) \cup \varphi^{2n/3}(\alpha)$, with α being the concatenation of two edges a and b, such that $b = \varphi^{n/3}(\bar{a})$, $\varphi^{n/3}(b) = \varphi^{2n/3}(\bar{a})$ and $\varphi^{2n/3}(b) = \bar{a}$ (see left side of Figure 19, where a_i and e_i stand for $\varphi^{in/3}(a)$ and $\varphi^{in/3}(e)$ respectively);*
3) *a set of $n/3$ trivalent vertices, when α reduces to a single vertex, that is the terminal edges it joins share a common trivalent end; in this case the whole graph Γ consists of a disjoint union $\widehat{T} \sqcup \varphi(\widehat{T}) \sqcup \ldots \sqcup \varphi^{n/3-1}(\widehat{T})$ of $n/3$ tripods, where $\widehat{T} = e \cup \varphi^{n/3}(e) \cup \varphi^{2n/3}(e)$ with α reduced to the trivalent vertex of $\widehat{T}$ (see right side of Figure 19, where e_i stands for the terminal edge $\varphi^{in/3}(e)$).*

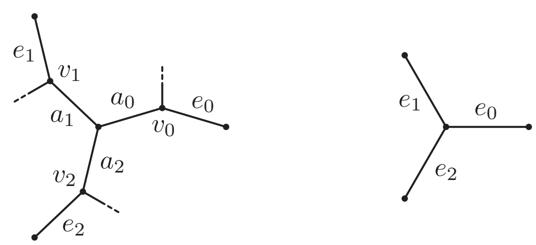

FIGURE 19. The form of the tripods T (left) and $\widehat{T}$ (right) in Lemma 4.2.

Proof. Let α be a path as in the statement and let $e \neq e'$ be the terminal edges it joins. Then the ends of α coincide with the unique trivalent ends v and v' of e and e' respectively.

If $v = v'$, that is $\mathrm{len}(\alpha) = 0$, then $\mathrm{ord}_\varphi(v) = n/3$ according to Lemma 2.2 and e' is either $\varphi^{n/3}(e)$ or $\varphi^{2n/3}(e)$. In any case, $\varphi^{n/3}$ cyclically permutes the three edges of the tripod $\widehat{T} = e \cup \varphi^{n/3}(e) \cup \varphi^{2n/3}(e)$, and we have the situation described in point 3 of the statement.

If $v \neq v'$, that is $\mathrm{len}(\alpha) \geq 1$, we consider any two distinct paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ with $i \neq j \bmod n$ and look once again at the four possible situations described in Lemmas 2.3 to 2.6.

As in the proof of Lemma 4.1, the situations of Lemmas 2.5 and 2.6 cannot occur. But here the argument to exclude the latter is different. Namely, if $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ were as in Lemma 2.6, then φ^{j-i} should swap the first edges of them starting from $\varphi^i(v) = \varphi^j(v)$, being $\varphi^i(e) = \varphi^j(e)$ the third edge at that vertex, and this would be in contrast with the oddness of n.

Therefore, $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ can meet only as in Lemma 2.4, and their common end $\varphi^j(v) = \varphi^i(v')$ cannot be shared by any other $\varphi^h(\alpha)$ (cf. proof of Lemma 3.1). Then, consider the two subpaths $\delta_{i,1} \subset \varphi^i(\alpha)$ and $\delta_{j,k} \subset \varphi^j(\alpha)$ in Figure 7.

If $\mathrm{len}(\delta_{i,1}) = \mathrm{len}(\delta_{j,k}) = \mathrm{len}(\alpha)/2$, then for the middle vertex u of α we have that $\varphi^i(u) = \varphi^j(u)$ is the common end of $\delta_{i,1}$ and $\delta_{j,k}$. By the same argument as above, this implies that $\mathrm{ord}_\varphi(u) = n/3$ and $\varphi^{n/3}$ cyclically permutes the three paths α, $\varphi^{n/3}(\alpha)$ and $\varphi^{2n/3}(\alpha)$. Moreover, such three paths are disjoint from any other path $\varphi^h(\alpha)$, hence we can conclude that $\cup_i \varphi^i(\alpha)$ is a disjoint union $T \sqcup \varphi(T) \sqcup \ldots \sqcup \varphi^{n/3-1}(T)$, with T as in the left side of Figure 20. Here, α_i stands for the path $\varphi^{in/3}(\alpha)$, which is the part of T in the corresponding circular sector, and any arc except the terminal ones represent a path of edges. Analogously, v_i and e_i stand for $\varphi^{in/3}(v)$ and $\varphi^{in/3}(e)$ respectively.

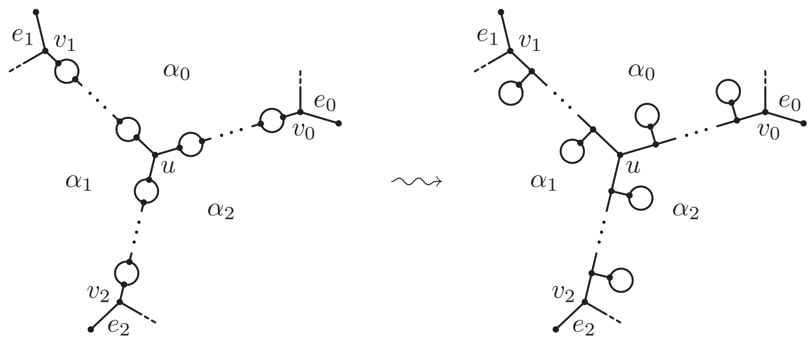

FIGURE 20. The starting configuration T in the proof of Lemma 4.1.

We can reduce any arc in this starting configuration T to a single edge, by performing φ-invariant slidings as in the proof of Lemma 3.2 (cf. Figure 15), in order to push all the edges attached along the α_i^s so that that they becomes attached to the last one exiting from v_i (and different from e_i). Then, the obvious φ-invariant F-moves allow us to change the resulting T into the form depicted on the right side of Figure 20, with each α_i being now the shortest path between v_i and $v_{i+1 \bmod 3}$. Finally, we can get the configuration described in point 2 of the statement and shown in the left side of Figure 19, by further φ-invariant slidings of the new edges attached along the actual α_i as above.

Otherwise, if $\mathrm{len}(\delta_{i,1}) = \mathrm{len}(\delta_{j,k}) > \mathrm{len}(\alpha)/2$, then the same argument as in the proof of Lemma 3.2 still works here, to put $\cup_i \varphi^i(\alpha)$ in the form stated in point 1 of the statement. $\square$

Proposition 4.3. *Let $\varphi \in \mathcal{A}_{g,b}$ be a non-trivial automorphism, whose order $n = \mathrm{ord}(\varphi) > 1$ is an odd multiple of 3. Then, φ is F-equivalent to the composition $\tau \circ \sigma$ of two automorphisms $\sigma, \tau \in \mathcal{A}_{g,b}$ such that $\mathrm{ord}(\sigma) = \mathrm{ord}(\tau) = 2$. Moreover, both σ and τ can be assumed to fix an edge and reverse an invariant edge.*

Proof. Given $\varphi : \Gamma \to \Gamma$ as in the statement, the same recursive construction of the proof of Proposition 3.3 will provide a sequence of automorphisms $\varphi_i : \Gamma_i \to \Gamma_i$ and a sequence of subgraphs $\Lambda_i \subset \Gamma_i$ with $i = 0, \dots, r$, satisfying the properties required in that proof, except for the following facts:

a) Λ_1 is allowed to be a subgraph of the first barycentric subdivision of Γ_1 instead that a subgraph of Γ_1 itself;

b) U_i is required to have cardinality n for $i = 1, \dots, r-1$ (Lemma 2.2 ensures that such cardinality is either n or $n/3$).

In particular, all the φ_i are F-equivalent to φ, and all the restrictions $\varphi_{i|\Lambda_i}$ admit a factorization $\tau_i \circ \sigma_i$ into two involutions of Λ_i. Hence, φ turns out to be F-equivalent to $\varphi_r = \tau_r \circ \sigma_r$ (being $\Lambda_r = \Gamma_r$), which proves the proposition.

The starting step of the recursion is provided by Lemma 4.1. If the situation described in point 1 of that lemma occurs, we define the automorphism $\varphi_1 : \Gamma_1 \to \Gamma_1$, the subgraph $\Lambda_1 \subset \Gamma_1$, the two involutions $\sigma_1, \tau_1 : \Lambda_1 \to \Lambda_1$ and the vertex u_1, as in the proof of Proposition 3.3. Otherwise, the situation described in point 2 of Lemma 4.1 occurs. In this case, there exists a φ-invariant disjoint union of $n/3$ tripods $T \sqcup \varphi(T) \sqcup \dots \sqcup \varphi^{n/3\,-\,1}(T) \subset \Gamma$, with $\varphi^{n/3}$ cyclically permuting the edges of the tripod T. Then, we put $\Gamma_1 = \Gamma_0 = \Gamma$, $\varphi_1 = \varphi_0 = \varphi$ and $\Lambda_1 = T' \sqcup \varphi(T') \sqcup \dots \sqcup \varphi^{n/3\,-\,1}(T') \subset \Gamma_1$, where $T' \subset T$ is the star of the trivalent vertex of T in the first barycentric subdivision. Moreover, we denote by v the trivalent vertex of T' and by u any free end of T', and we define σ_1 and τ_1 by putting $\sigma_1(\varphi^j(v)) = \varphi^{-j}(v)$, $\sigma_1(\varphi^j(u)) = \varphi^{-j}(u)$, $\tau_1(\varphi^j(v)) = \varphi^{1-j}(v)$ and $\tau_1(\varphi^j(u)) = \varphi^{1-j}(u)$ for $j = 0, \dots, n-1$.

In order to conclude the starting step, it is enough to observe that: σ_1 and τ_1 are well-defined involutive automorphisms of Λ_1, since they preserve the adjacency of vertices, being $\varphi^j(v)$ adjacent to $\varphi^k(u)$ if and only if $j = k \bmod n/3$; $\tau_1 \circ \sigma_1$ coincides with the restriction $\varphi_{1|\Lambda_1}$; $\Delta_1 = \mathrm{Cl}(\Gamma_1 - \Lambda_1)$ is a uni/trivalent graph; $U_1 = \Delta_1 \cap \Lambda_1$ consists of the φ_1-orbit of the univalent vertex u of Δ_1 and it has cardinality n; $u_1 = u$ is the only vertex of U_1 fixed by σ_1.

Now, to realize the recursive step of the construction, assume we are given $\varphi_{i-1} : \Gamma_{i-1} \to \Gamma_{i-1}$, $u_{i-1} \in U_{i-1} \subset \Lambda_{i-1} \subsetneq \Gamma_{i-1}$ and $\sigma_{i-1}, \tau_{i-1} : \Lambda_{i-1} \to \Lambda_{i-1}$ with the properties required in the proof of Proposition 3.3 for $i - 1 < r$, integrated by the points a and b said above. In particular, the φ_{i-1}-orbit U_{i-1} has cardinality n. Therefore, since $\mathrm{ord}(\varphi_{i-1}) = n$ by Lemma 2.1, also the restriction of φ_{i-1} to the uni/trivalent graph $\Delta_{i-1} = \mathrm{Cl}(\Gamma_{i-1} - \Lambda_{i-1})$ is an automorphism of the same order n. Furthermore, the terminal edges of Δ_{i-1} ending at vertices in U_{i-1} are all distinct (see proof of Proposition 3.3).

If Δ_{i-1} does not contain any path joining different terminal edges ending at vertices in U_{i-1}, then we can terminate the recursion with the i-th step, in the same way as in the proof of Proposition 3.3.

Otherwise, if a path $\alpha \subset \Delta_{i-1}$ exists joining different terminal edges of Δ_{i-1} ending at vertices in U_{i-1}, then we can choose it to be minimal (hence simple) and apply Lemma 4.2 to the restriction of φ_{i-1} to Δ_{i-1} and to such a minimal α, in order to get the structure described in that statement for $\cup_j \varphi_{i-1}^j(\alpha) \subset \Delta_{i-1}$ up to F-equivalence. Such an F-equivalence only involves internal edges of Δ_{i-1}, hence it does not change the set of free ends $U_{i-1} \subset \Delta_{i-1}$ and the restriction of $\varphi_{i-1|\Delta_{i-1}}$. Therefore, it can be extended to an F-equivalence of the whole φ_{i-1} on Γ_{i-1}, which leaves Λ_{i-1} and the restriction $\varphi_{i-1|\Lambda_{i-1}}$ unchanged.

As a result, we get a new uni/trivalent graph Γ_i, such that $\Lambda_{i-1} \subset \Gamma_i$ and a new automorphism $\varphi_i : \Gamma_i \to \Gamma_i$, which is F-equivalent to φ_{i-1} and coincides with φ_{i-1} on Λ_{i-1}. We also get a new minimal path $\alpha \subset \mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$, which joins different φ_i-equivalent terminal edges of $\mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$ ending at vertices in U_{i-1}, such that $\cup_j \varphi_i^j(\alpha)$ itself (no more up to F-equivalence) has the structure stated in Lemma 4.2. Then, we proceed by distinguishing the different situations described in points 1 to 3 of that lemma.

If the situation of point 1 occurs, then the i-th step is the same as in the proof of Proposition 3.3. In particular, if such step is not the concluding one, then U_i can be easily seen to have cardinality n.

In the case when the situation of point 2 occurs, we denote by $e \subset \Gamma_i$ the terminal edge of $\mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$ whose free end is u_{i-1}. Then, we choose u_i to be the other end of e and put $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_i^j(e) \cup_j \varphi_i^j(T)$, in such a way that $\Delta_i = \Gamma_i - \Lambda_i$ is a uni/trivalent graph, which meets Λ_i at the φ_i-orbit U_i of u_i. Moreover, we define σ_i and τ_i to be the unique automorphisms of Λ_i extending σ_{i-1} and τ_{i-1} respectively. The existence of such extensions and all the required properties of them, in particular the equality $\tau_i \circ \sigma_i = \varphi_{i|\Lambda_i}$ and the fact that u_i is the unique vertex of U_i fixed by σ_i, can be proved by the same argument as in the proof of Lemma 3.3.

Finally, if the situation of point 3 occurs, we just put $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_i^j(\widehat{T}) = \Gamma_i$, where the last equality is due to the connectedness of Γ_i, and define σ_i and τ_i to be the unique extensions of σ_{i-1} and τ_{i_1} to Γ_i. These can be easily seen to verify all the required properties for $i = r$, hence the recursion terminates with this step. □

5. The case of order 2^m

In order to deal with automorphisms $\varphi : \Gamma \to \Gamma$ in $\mathcal{A}_{g,b}$ of order 2^m, we first prove that we can limit ourselves to consider the case when $\mathrm{ord}_\varphi(e) = 2^m$ for every edge e of Γ (hence, according to Lemma 2.2, $\mathrm{ord}_\varphi(v) = 2^m$ for every vertex v of Γ).

Lemma 5.1. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism in $\mathcal{A}_{g,b}$ with $\mathrm{ord}(\varphi) = 2^m$, and let $2^k = \min_{e \in \Gamma} \mathrm{ord}_\varphi(e)$, where e varies among all the edges of Γ. Then, up to F-equivalence and composition with elementary automorphisms, φ can be reduced to an automorphism $\psi : \Gamma' \to \Gamma'$ such that $\mathrm{ord}(\psi) = 2^k$ and $\mathrm{ord}_\psi(e) = 2^k$ for every edge e of Γ'.*

Proof. The proof is by induction on the pair (m, n_m) with respect to the lexicographic order, where n_m is the number of the edges e of Γ such that $\mathrm{ord}_\varphi(e) = 2^m$.

The base of the induction is for $m = k$, and hence $\mathrm{ord}_\varphi(e) = 2^m$ for all the edges e of Γ, that is $n_m = 3g - 3 + 2b$.

If on the contrary $m > k$, then Lemma 2.2 tells us that there are three distinct edges e_0, e_1 and e_2 of Γ sharing a trivalent vertex v of Γ, such that $\mathrm{ord}_\varphi(e_0) = \mathrm{ord}_\varphi(v) = 2^{m-1}$, $\mathrm{ord}_\varphi(e_1) = \mathrm{ord}_\varphi(e_2) = 2^m$ and $e_2 = \varphi^{2^{m-1}}(e_1)$ (note that $e_1 \neq e_2$, otherwise $e_1 = e_2$ would be a loop and its order should be 2^{m-1}). Let v_1 and v_2 the other ends of e_1 and e_2 respectively.

If v_1 and v_2 are (distinct) univalent vertices, then by composing with the terminal switch S_{e_1,e_2} we get an automorphism $\rho = S_{e_1,e_2} \circ \varphi$ with either $m(\rho) < m(\varphi)$ or $m(\rho) = m(\varphi)$ and $n_m(\rho) < n_m(\varphi)$. In fact, $\mathrm{ord}_\rho(e) = \mathrm{ord}_\varphi(e)/2$ for all the edges e in the φ-orbit of e_1 (and e_2), while $\mathrm{ord}_\rho(e) = \mathrm{ord}_\varphi(e)$ for all the other edges of Γ.

If v_1 and v_2 are coinciding trivalent vertices, then the same argument as above applies, but with S_{e_1,e_2} an internal switch.

Finally, assume that v_1 and v_2 are distinct trivalent vertices. Then, we have the situation depicted in the left side of Figure 21, where $\varphi^{2^{m-1}}(a_i) = a_{3-i}$ and $\varphi^{2^{m-1}}(b_i) = b_{3-i}$ for $i = 1, 2$ (with the four edges a_1, a_2, b_1 and b_2 not necessarily distinct). In this case, we can perform the φ-invariant move $F_{\mathcal{T},\mathcal{T}'}$ acting on a neigh-

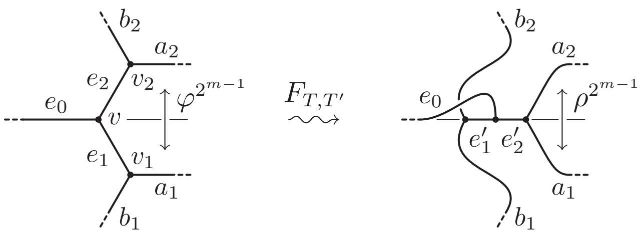

FIGURE 21. Reducing $(m(\varphi), n_m(\varphi))$.

borhood of $e_1 \cup e_2$ as described in Figure 21. This changes φ into an automorphism ρ with either $m(\rho) < m(\varphi)$ or $m(\rho) = m(\varphi)$ and $n_m(\rho) < n_m(\varphi)$. In fact $F_{\mathcal{T},\mathcal{T}'}$, replaces the φ-orbit of e_1 (and e_2) by two different ρ-orbits of e_1' and e_2' respectively, having cardinality 2^{m-1}. This completes proof of the inductive step. □

In the light of the previous lemma, the next two lemmas concern the special case of an automorphism $\varphi : \Gamma \to \Gamma$ with $\mathrm{ord}(\varphi) = n = 2^m$ and $\mathrm{ord}_\varphi(e) = n$ for every edge e of Γ. We notice that, under the latter assumption, the situation described in Lemma 2.6 can occur only with $\varphi^i(\alpha) = \varphi^j(\alpha)$, that is $i = j \bmod n$.

Lemma 5.2. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a (possibly disconnected) uni/trivalent graph Γ, with $\mathrm{ord}(\varphi) = n = 2^m$ and $\mathrm{ord}_\varphi(e) = n$ for every edge e of Γ. If $\alpha \subset \Gamma$ is a (simple) path of minimal length among all paths joining any two distinct φ-equivalent vertices, then $\cup_i \varphi^i(\alpha)$ is one of the following:*

1) *a disjoint union $C \sqcup \varphi(C) \sqcup \ldots \sqcup \varphi^{n/\ell-1}(C)$ of n/ℓ simple cycles (possibly a single one, for $\ell = n$), each given by a concatenation of images of α, having the same structure as described in Lemma 3.1; in this case, up to F-equivalence we can assume α to consist of a single edge a (see Figure 22, where ℓ is a power of 2, a_i stands for $\varphi^{is}(a)$ and a similar notation is adopted for the vertices v_i as well);*

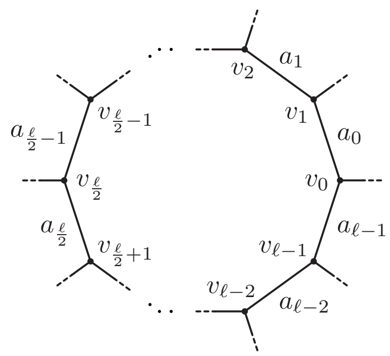

FIGURE 22. The form of the cycle C in Lemma 5.2.

2) *a disjoint union $D \sqcup \varphi(D) \sqcup \ldots \sqcup \varphi^{n/2-1}(D)$ of $n/2$ "diagonal" edges (possibly a single one, for $n = 2$), where $D = \alpha$ and $\varphi^{n/2}(D) = \overline{D}$ (see Figure 23, where the vertices v_0 and v_1 are switched by $\varphi^{n/2}$, and they can be either both univalent as on the left side or both trivalent as on the right side).*

FIGURE 23. The "diagonal" edge D in Lemma 5.2.

Proof. Let $\alpha \subset \Gamma$ be an arbitrary path of minimal length among all paths joining any two distinct φ-equivalent vertices and let $v \neq v'$ be its ends. For any $i \neq j \bmod n$, let us consider the two paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$, and the four different situations described in Lemmas 2.3 to 2.6, which cover all the possibilities, thanks to the φ-equivalence of v and v'.

The situation of Lemma 2.6 cannot occur, due to the choice of i and j, while any of the three remaining situations may occur. In particular, $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ can have non-empty intersection both as in Lemma 2.5 and as in Lemma 2.4.

Let us assume first that $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ are as in Lemma 2.5, i.e. they share both ends, with $\varphi^i(v) = \varphi^j(v')$ and $\varphi^j(v) = \varphi^i(v')$. In particular, φ^{j-i} switches v and v', and $j - i = n/2 \bmod n$, since $\operatorname{ord}_\varphi(v) = n$.

The global minimality of α ensures that either α is a single "diagonal" edge D with $\varphi^{n/2}(D) = \overline{D}$, or α and $\varphi^{n/2}(\alpha)$ only meet at their common ends v and v'. Moreover, α and $\varphi^{n/2}(\alpha)$ are disjoint from any other $\varphi^k(\alpha)$ with $k \neq 0, n/2 \bmod n$, thanks to Lemma 2.3. If α is a single "diagonal" edge D, we have that $\cup_i \varphi^i(\alpha)$ is as in point 2 of the statement. Otherwise, $\cup_i \varphi^i(\alpha)$ consists in the disjoint union $C \sqcup \varphi(C) \sqcup \ldots \sqcup \varphi^{n/2-1}(C)$ of $n/2$ simple cycles (possibly a single one, for $n = 2$), with $C = \alpha \cup \varphi^{n/2}(\alpha)$. Then, arguing as in the proof of Lemma 3.1, the length of α (and that of its images as well) may be reduced to 1, which gives a special case of the situation described in point 1 of the statement.

At this point, we are left with the case when $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ can only meet as in Lemma 2.4. In this case, denoting by $\varphi^i(u)$ with $u \in \alpha$ the common end of $\delta_{i,1}$ and $\delta_{j,k}$, we have that $\varphi^j(u) \in \delta_{j,k}$ since $\operatorname{len}(\delta_{i,1}) = \operatorname{len}(\delta_{j,k}) \geq \operatorname{len}(\alpha)/2$. Actually, $\varphi^j(u)$ has to coincide with either $\varphi^j(v')$ or $\varphi^i(u)$, otherwise the global minimality of α would be contradicted. However, $\varphi^j(u) = \varphi^i(u)$ is not possible because this would imply that $\operatorname{ord}_\varphi(u) < n$. Thus, $\varphi^j(u) = \varphi^j(v')$ and we can conclude that the set $\cup_i \varphi^i(\alpha)$ has the structure described in point 1 of the statement, by arguing as in the proof of Lemma 3.1. □

Lemma 5.3. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism of a (possibly disconnected) uni/trivalent graph Γ, with $\operatorname{ord}(\varphi) = n = 2^m$ and $\operatorname{ord}_\varphi(e) = n$ for every edge e of Γ. If $\alpha \subset \Gamma$ is a (simple) path of minimal length among all paths joining any two distinct terminal edges in a given φ-orbit, then up to F-equivalence we can assume $\cup_i \varphi^i(\alpha)$ to be one of the following:*

1) *of a disjoint union $C \sqcup \varphi(C) \sqcup \ldots \sqcup \varphi^{n/\ell-1}(C)$ of n/ℓ simple cycles (possibly a single one for $\ell = n$), each given by a concatenation of images of α, having the same structure as described in Lemma 3.2, with α consisting of either one edge a or two edges a and b (see Figure 24, where ℓ is a power of 2, a_i and b_i stand for $\varphi^{is}(a)$ and $\varphi^{is}(b)$ respectively, and a similar notation is adopted for the terminal edges e_i's and the vertices v_i's as well);*

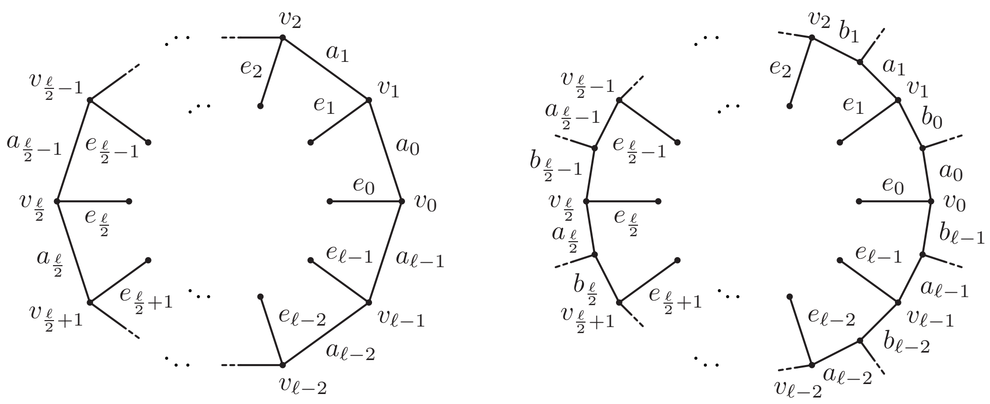

FIGURE 24. The form of the cycle C in Lemma 5.3.

2) *a disjoint union $D \sqcup \varphi(D) \sqcup \ldots \sqcup \varphi^{n/2-1}(D)$ of $n/2$ "diagonal" edges (possibly a single one, for $n = 2$), where $D = \alpha$ and $\varphi^{n/2}(D) = \overline{D}$ (see Figure 25, where the edges e_0 and e_1 are switched by $\varphi^{n/2}$).*

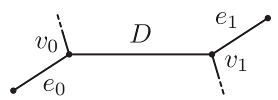

FIGURE 25. The "diagonal" edge D in Lemma 5.3.

Proof. Let α be a path as in the statement, and let $e \neq e'$ be the terminal edges it joins. Then the ends of α coincide with the unique trivalent ends v and v' of e and e' respectively. Notice that $v \neq v'$ since otherwise $\operatorname{ord}_\varphi(v)$ would be less than n.

For $i \neq j \mod n$, the two paths $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ can meet both as in Lemma 2.4 and as in Lemma 2.5, but not as in Lemma 2.6.

Let us assume first that $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ are as in Lemma 2.5, i.e. they share both ends, with $\varphi^i(v) = \varphi^j(v')$ and $\varphi^j(v) = \varphi^i(v')$, and their union $\varphi^i(\alpha) \cup \varphi^j(\alpha)$ looks like in Figure 9. In particular, φ^{j-i} switches v and v', and then $j - i = n/2 \mod n$, since $\mathrm{ord}_\varphi(v) = n$. Moreover, Lemma 2.3 ensures that $\varphi^i(\alpha) \cup \varphi^j(\alpha)$ is disjoint from any other path $\varphi^k(\alpha)$ with $k \neq i, j \bmod n$.

Hence, $\cup_i \varphi^i(\alpha)$ is a disjoint union of $n/2$ pairs of "diagonal" paths, each pair being given by $\varphi^i(\alpha)$ and $\varphi^{i+n/2}(\alpha)$ for some $i = 0, \dots, n/2-1$. Figure 26 shows the two possible forms of $\varphi^i(\alpha) \cup \varphi^{i+n/2}(\alpha)$, depending on the fact that α and $\varphi^{n/2}(\alpha)$ start and end with non-trivial common subpaths (left side) or not (right side). Here, the subscripts denote the images under the corresponding power of φ, and apart from the edges e_i and $e_{i+n/2}$ each arc represents a path of edges. We notice that in both cases, if the length of the path α is even, then the central vertices of the paths α_i and $\alpha_{i+n/2}$ have to be distinct (that is, they have to belong to non-common subarcs), otherwise the order of such vertices would be $n/2$.

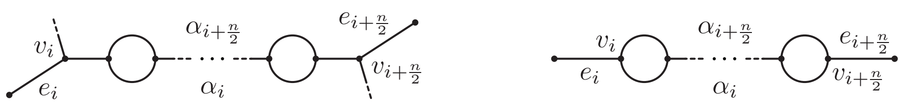

FIGURE 26. The starting form of $\varphi^i(\alpha) \cup \varphi^{i+n/2}(\alpha)$ in the proof of Lemma 5.3.

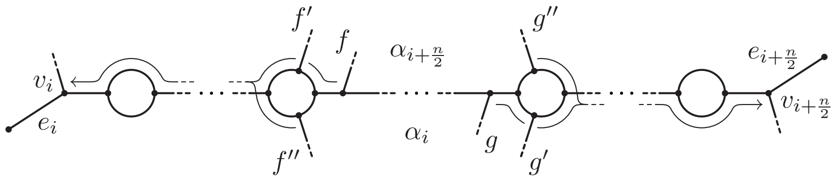

FIGURE 27. Simplifying the configuration on the left side of Figure 26.

The configuration on the left side of Figure 26 can be simplified by a sequence of symmetric pairs of φ-invariant slidings as indicated in Figure 27 (cf. Figure 15), in order to reduce all the arcs to single edges. After that, all the resulting bigons, except the central one if they are odd in number, can be modified in pairs as in Figure 20 and then slided out of $\varphi^i(\alpha) \cup \varphi^{i+n/2}(\alpha)$ as above. Then, in the case of an even number of bigons we end up with the situation described in point 1 of the statement. Otherwise, we are left with one bigon as shown on the left side of Figure 28, and we can perform the F-move described in that figure, to get the configuration on the right side of the same Figure 28. This is the special case for $\ell = 2$ of the configuration on the right side of Figure 24 considered in point 1 of the statement.

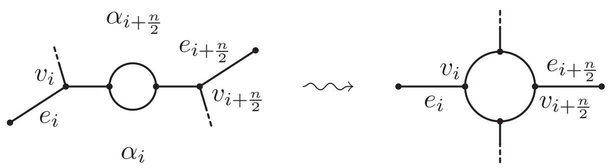

FIGURE 28. Simplified version of the configuration on the left side of Figure 26.

Now, look at the configuration on the right side of Figure 26. If $\mathrm{len}(\alpha) = 1$, that is $\varphi^i(\alpha) \cup \varphi^{i+n/2}(\alpha)$ consists of a single bigon, then we just have the special case for $\ell = 2$ of the configuration on the left side of Figure 24 considered in point 1 of the statement. If $\mathrm{len}(\alpha) > 1$, then we can simplify the configuration by the same argument as above, but sliding everything to the most external loops adjacent to e_i and $e_{i+n/2}$. The result is shown on the left side of Figure 29. We can change it into the configuration on the right side of the figure, by a φ-invariant F-move on the central edge, and then to the configuration on the right side of Figure 28, by a further φ-invariant sliding (reducing the length of α to 2). So, we get once again a special case of the situation described in point 1 of the statement.

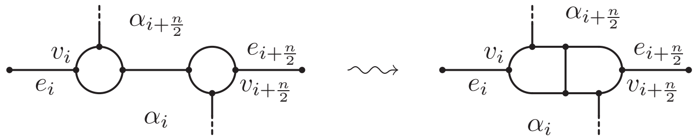

FIGURE 29. Simplified version of the configuration on the right side of Figure 26.

At this point, we are left with the case when $\varphi^i(\alpha)$ and $\varphi^j(\alpha)$ can only meet as in Lemma 2.4. In this case, then the fact that n is even ensures that their common end $\varphi^j(v) = \varphi^i(v')$ cannot be shared by any other $\varphi^h(\alpha)$ and that $\mathrm{len}(\delta_{i,1}) = \mathrm{len}(\delta_{j,k}) > \mathrm{len}(\alpha)/2$ (cf. Figure 7). Therefore, the same arguments as in Lemma 3.1 ensure that the situation can be reduced to the one described in point 1 of the statement. □

Proposition 5.4. *Let $\varphi : \Gamma \to \Gamma$ be a non-trivial automorphism in $\mathcal{A}_{g,b}$, with $\mathrm{ord}(\varphi) = n = 2^m$. Then, up to F-equivalence and composition with elementary automorphisms, we can reduce φ to the composition $\tau \circ \sigma$ of two automorphisms $\sigma, \tau \in \mathcal{A}_{g,b}$ such that $\mathrm{ord}(\sigma) = \mathrm{ord}(\tau) = 2$. Moreover, σ can be assumed to fix an edge and τ can be assumed to reverse an invariant edge.*

Proof. Given $\varphi : \Gamma \to \Gamma$ as in the statement, thanks to Lemma 5.1 we can assume that $\mathrm{ord}_\varphi(e) = n$ for every edge e of Γ. Then, the same recursive construction of the proof of Proposition 3.3 will provide a sequence of automorphisms $\varphi_i : \Gamma_i \to \Gamma_i$ and a sequence of subgraphs $\Lambda_i \subset \Gamma_i$ with $i = 0, \dots, r$, satisfying the properties required in that proof, except for the following facts:

a) Λ_1 is allowed to be a subgraph of the second barycentric subdivision of Γ_1 instead that a subgraph of Γ_1 itself;
b) $\mathrm{ord}_{\varphi_i}(e) = n$ for every edge e of Γ_i and every $i = 1, \dots, r$;
c) σ_i fixes exactly two vertices u_i and $\varphi_i^{n/2}(u_i)$ in U_i, for every $i = 0, \dots, r-1$;
d) σ_i fixes two edges of Γ_i, but it is not required to reverse any invariant edge, for every $2 \leq i \leq r$;
e) the τ_i's are not required to fix or reverse any edge, except for τ_r (the last one) that has to reverse an invariant edge.

In particular, all the φ_i's are F-equivalent to φ, and all the restrictions $\varphi_{i|\Lambda_i}$ admit a factorization $\tau_i \circ \sigma_i$ into two involutions of Λ_i. Hence, φ turns out to be F-equivalent to $\varphi_r = \tau_r \circ \sigma_r$ (being $\Lambda_r = \Gamma_r$), which proves the proposition.

The starting step of the recursion is provided by Lemma 5.2. If the situation described in point 1 of that lemma occurs, we define the automorphism $\varphi_1 : \Gamma_1 \to \Gamma_1$, the subgraph $\Lambda_1 \subset \Gamma_1$, the two involutions $\sigma_1, \tau_1 : \Lambda_1 \to \Lambda_1$ and the vertex u_1, as in the proof of Proposition 3.3. Otherwise, the situation described in point 2 of Lemma 5.2 occurs. In this case, there exists a φ-invariant disjoint union of $n/2$ "diagonal" edges $D \sqcup \varphi(D) \sqcup \ldots \sqcup \varphi^{n/2\,-1}(D) \subset \Gamma$, with $\varphi^{n/2}(D) = \overline{D}$. Then, we put $\Gamma_1 = \Gamma_0 = \Gamma$, $\varphi_1 = \varphi_0 = \varphi$ and $\Lambda_1 = D' \sqcup \varphi(D') \sqcup \ldots \sqcup \varphi^{n/2\,-1}(D') \subset \Gamma_1$, where $D' \subset D$ is the start of the barycenter of D in the second barycentric subdivision of Γ. Moreover, we denote by u any one of the two ends of D', and we define σ_1 and τ_1 by putting $\sigma_1(\varphi^j(u)) = \varphi^{-j}(u)$ and $\tau_1(\varphi^j(u)) = \varphi^{1-j}(u)$ for $j = 0, \ldots, n-1$.

In order to conclude the starting step, it is enough to observe that: $\mathrm{ord}_{\varphi_1}(e) = n$ for every edge e of Γ_1; σ_1 and τ_1 are well-defined involutive automorphisms of Λ_1, since they preserve the adjacency of vertices, being $\varphi^j(u)$ adjacent to $\varphi^k(u)$ if and only if $j = k \bmod n/2$; $\tau_1 \circ \sigma_1$ coincides with the restriction $\varphi_{1|\Lambda_1}$; $\Delta_1 = \mathrm{Cl}(\Gamma_1 - \Lambda_1)$ is a uni/trivalent graph; $U_1 = \Delta_1 \cap \Lambda_1$ consists of the φ_1-orbit of the univalent vertex u of Δ_1 and it has cardinality n; $u_1 = u$ and $\varphi^{n/2}(u_1)$ are the only vertices of U_1 fixed by σ_1.

Now, to realize the recursive step of the construction, assume we are given $\varphi_{i-1} : \Gamma_{i-1} \to \Gamma_{i-1}$, $u_{i-1} \in U_{i-1} \subset \Lambda_{i-1} \subsetneq \Gamma_{i-1}$ and $\sigma_{i-1}, \tau_{i-1} : \Lambda_{i-1} \to \Lambda_{i-1}$ with the properties required in the proof of Proposition 3.3 for $i - 1 < r$, integrated by the points a to e said above. In particular, the restriction of φ_{i-1} to the uni/trivalent graph $\Delta_{i-1} = \mathrm{Cl}(\Gamma_{i-1} - \Lambda_{i-1})$ is an automorphism of order n, and the order of each edge of Δ_{i-1} with respect to it is n as well.

If two of the terminal edges of Δ_{i-1} ending at vertices in U_{i-1} coincide, then there are $n/2$ such terminal edges and each of them is a "diagonal" edge reversed by $\varphi_{i-1}^{n/2}$. In this case, denoting by D the "diagonal" edge between u_{i-1} and $\varphi_{i-1}^{n/2}(u_{i-1})$, we put $\Gamma_i = \Gamma_{i-1}$, $\varphi_i = \varphi_{i-1}$ and $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_{i-1}^j(D)$, where the last equality is due to the connectedness of Γ_i. Moreover, we define σ_i and τ_i to be the unique extensions of σ_{i-1} and τ_{i-1} to Γ_i. These can be easily seen to exist and to verify all the required properties for $i = r$, except for the fact that τ_r reverses some invariant edge, which will be proved at the end of the proof. Hence, this step terminates the recursion.

Then, let the terminal edges of Δ_{i-1} ending at vertices in U_{i-1} be all distinct.

If Δ_{i-1} does not contain any path joining two different such terminal edges, then we can terminate the recursion with the i-th step, in the same way as in the proof of Proposition 3.3. Once again, we postpone at the end of the proof the verification that τ_r reverses some invariant edge.

Otherwise, if a path $\alpha \subset \Delta_{i-1}$ exists joining different terminal edges of Δ_{i-1} ending at vertices in U_{i-1}, then we can choose it to be minimal (hence simple) and apply Lemma 5.3 to the restriction of φ_{i-1} to Δ_{i-1} and to such a minimal α, in order to get the structure described in that statement for $\cup_j \varphi_{i-1}^j(\alpha) \subset \Delta_{i-1}$ up to F-equivalence. Such an F-equivalence only involves internal edges of Δ_{i-1}, hence it does not change the set of free ends $U_{i-1} \subset \Delta_{i-1}$ and the restriction of $\varphi_{i-1|\Delta_{i-1}}$. Therefore, it can be extended to an F-equivalence of the whole φ_{i-1} on Γ_{i-1}, which leaves Λ_{i-1} and the restriction $\varphi_{i-1|\Lambda_{i-1}}$ unchanged.

As a result, we get a new uni/trivalent graph Γ_i, such that $\Lambda_{i-1} \subset \Gamma_i$ and a new automorphism $\varphi_i : \Gamma_i \to \Gamma_i$, which is F-equivalent to φ_{i-1} and coincides with φ_{i-1} on Λ_{i-1}. We also get a new minimal path $\alpha \subset \mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$, which joins different φ_i-equivalent terminal edges of $\mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$ ending at vertices in U_{i-1}, such that $\cup_j \varphi_i^j(\alpha)$ itself (no more up to F-equivalence) has the structure stated in Lemma 5.3.

Then, we proceed by distinguishing the different situations described in points 1 and 2 of that lemma.

If the situation of point 1 occurs, then the i-th step is the same as in the proof of Proposition 3.3, but now σ_i fixes two edges and τ_i reverses two invariant edges.

In the case when the situation of point 2 occurs, we denote by e the terminal edge of $\mathrm{Cl}(\Gamma_i - \Lambda_{i-1})$ whose free end is u_{i-1}. Then, we choose u_i to be the other end of e and put $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_i^j(e) \cup_j \varphi_i^j(D)$, in such a way that $\Delta_i = \mathrm{Cl}(\Gamma_i - \Lambda_i)$ is a uni/trivalent graph, which meets Λ_i at the φ_i-orbit U_i of u_i. Moreover, we define σ_i and τ_i to be the unique automorphisms of Λ_i extending σ_{i-1} and τ_{i-1} respectively. To see that such extensions exist, we first define $\sigma_i(\varphi_i^j)(u_i) = \varphi_i^{-j}(u_i)$ and $\tau_i(\varphi_i^j)(u_i) = \varphi_i^{1-j}(u_i)$, for every $j = 0, \dots, n-1$, and then we verify that these definitions, together with the previous definitions of σ_{i-1} and τ_{i-1}, preserve adjacency of vertices in Λ_i. Notice that $\tau_i \circ \sigma_i$ trivially coincides with $\varphi_{i|\Lambda_i}$. Furthermore, σ_i fixes only the vertices u_i and $\varphi_i^{n/2}(u_i)$ in the φ_i-orbit U_i, and it fixes the edges e and $\varphi^{n/2}(e)$. On the contrary, τ_i neither fixes nor reverses any edge in the φ_i-orbits of e and D. This conclude the recursive step.

Finally, we observe that the situation of point 1 of Lemma 5.3 occurs in at least one of the recursive steps due to the connectedness of Γ, and from that step on all the τ_i reverse some invariant edge. In particular, this is true for τ_r. $\square$

At this point, taking into account Propositions 3.3, 4.3 and 5.4, we are left to show that any automorphism in $\mathcal{A}_{g,b}$ belongs to $\mathcal{E}_{g,b}$, in the special case when it has order 2 and fixes or reverses some edge. This is the content of our last proposition.

Proposition 5.5. *Let $\varphi : \Gamma \to \Gamma$ be an automorphism in $\mathcal{A}_{g,b}$ of order 2, which either fixes an edge or reverses an invariant edge. Then, up to F-equivalence and composition with elementary switches, we can reduce φ to the identity.*

Proof. If φ fixes an edge e, then Lemma 5.1 ensures that up to F-equivalence and composition with elementary automorphisms, φ can be reduced to an automorphism $\psi : \Gamma' \to \Gamma'$ such that $\mathrm{ord}(\psi) = 1$, that is ψ reduces to the identity. On the other hand, if φ reverses an invariant edge e, we can perform a φ-invariant F-move $F_{e,e'}$, in order to get a new automorphism $\varphi' : \Gamma' \to \Gamma'$, which fixes the edge e' of Γ'. $\square$

Appendix

An infinite simply connected 2-dimensional complex $\mathcal{R}_{g,b}$, codifying all the pant decompositions on the connected compact oriented surface $\Sigma_{g,b}$ of genus g with b boundary components, as well as all the moves relating them and all the relations between those moves, was introduced by Moore and Seiberg in [14]. Successively, $\mathcal{R}_{g,b}$ it has been studied, by using the Cerf theoretic techniques introduced in [13], in [10, 12] and [2].

In [3], we built up a different version of the complex $\mathcal{R}_{g,b}$ and studied it by following the original Moore and Seiberg's approach. Namely, our construction of $\mathcal{R}_{g,b}$ is subdivided into two independent steps.

First a finite simply connected 2-dimensional complex $\mathcal{S}_{g,b}$ is described, which codifies the *combinatorial structures* of pant decompositions of $\Sigma_{g,b}$. The combinatorial structure of a pant decomposition D consists of the information concerning only the incidence relations between pants, and it is encoded by the dual graph $\Gamma_D \in \mathcal{G}_{g,b}$, with the trivalent vertices corresponding to the pants, the univalent vertices corresponding to the boundary components, and the edges corresponding to the cutting curves. Afterwards, by using a presentation of the mapping class group $\mathcal{M}_{g,b}$ of $\Sigma_{g,b}$, we get the desired complex $\mathcal{R}_{g,b}$.

Due to technical issues, the switch from $\mathcal{S}_{g,b}$ to $\mathcal{R}_{g,b}$ is not a "direct" one, but it requires instead a cumbersome intermediate step. This involves in the construction of two additional complexes $\widetilde{\mathcal{S}}_{g,b}$ and $\widetilde{\mathcal{R}}_{g,b}$, codifying respectively the *decorated combinatorial structures* and the *decorated pant decompositions*. Here, a decoration of a pant decomposition D is a numbering of its cutting curves, and a decoration of the corresponding combinatorial structure is a numbering of the edges of Γ_D.

This method allows us to sidestep a crucial problem in the direct transition from $\mathcal{S}_{g,b}$ to $\mathcal{R}_{g,b}$, that is the study of the stabilization subgroup $\mathrm{Stab}(D) \in \mathcal{M}_{g,b}$ of any given pant decomposition D. In fact, the proof of the simple connectedness of $\mathcal{R}_{g,b}$ given in [3] would become less elaborate based on the fact that the fiber $p^{-1}(\Gamma_D)$ of the natural projection $p : \mathcal{R}_{g,b} \to \mathcal{S}_{g,b}$ is trivial at $\mathcal{R}_{g,b}$ on the level of the fundamental group. In other words, we need to show that any loop based at D and contained in $p^{-1}(\Gamma_D)$ is contractible in $\mathcal{R}_{g,b}$. Now, such a loop corresponds to an element of $\mathrm{Stab}(D)$, that is a symmetry of D, which induces a (possibly trivial) combinatorial symmetry of Γ_D. Actually, once the combinatorial symmetries of Γ_D are known, one can reconstruct those of D in a straightforward way, by adding some easy topological information.

At this point, it should be clear the motivation for the study carried out in the present paper of the structure of $\mathcal{A}_{g,b}$, and in particular of the inclusion $\mathcal{A}(\Gamma) \subset \mathcal{A}_{g,b}$ of the group of combinatorial symmetries $\mathcal{A}(\Gamma)$ of a given $\Gamma \in \mathcal{G}_{g,b}$. In fact, the result we obtain here will enable us to give a much simpler construction of $\mathcal{R}_{g,b}$ than in [3], and a more direct proof of its simple connectedness starting from $\mathcal{S}_{g,b}$ without involving decorations.

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