BRANCHED COVERINGS OF SIMPLY CONNECTED 4-MANIFOLDS

VALENTINA BAIS, RICCARDO PIERGALLINI, AND DANIELE ZUDDAS

Abstract. We show that, given $d \geqslant 4$ and two closed connected oriented PL 4-manifolds M and N such that N has a handle decomposition with no 1- and 3-handles, there exists a d-fold simple branched covering $p\colon M \xrightarrow{d:1} N$ if and only if there is an isometric embedding of intersection lattices $d \cdot I_N \hookrightarrow I_M$. Moreover, if such p exists, one can build it in such a way that its branch set $B_p \subset N$ is locally flat PL embedded if $d \geqslant 5$ and has at most nodal singularities if $d = 4$.

1. Introduction

By a classical result of Alexander [1], any closed connected oriented PL n-manifold M admits a branched covering

$$p\colon M \xrightarrow{d:1} S^n$$

of some degree $d \geqslant 1$. One possible interpretation of this fact is that any such a manifold M can be presented by means of a codimension two closed subpolyhedron $B_p \subset S^n$, i.e. the branch set, and a group homomorphism $\omega_p\colon \pi_1(S^n \setminus B_p) \to S_d$, i.e. the monodromy representation of the unbranched part of p, where S_d denotes the permutation group of d elements. We remark such presentation is not optimal, since the degree d depends on the cardinality of a chosen triangulation of M and the branch set B_p is not a PL submanifold. However, the following improvements of Alexander's result in low dimensions are known. The easiest case is when $M = \Sigma_g$ is a closed oriented surface of genus g: one gets a simple d-fold branched covering $\Sigma_g \xrightarrow{d:1} S^2$ for any degree $d \geqslant 2$ by stabilizing the quotient map induced by the hyperelliptic involution. Moving to the case $n = 3$, Hilden [12], Hirsch [13] and Montesinos [18] independently proved that any closed oriented 3-manifold can be represented by means of a 3-fold dihedral covering of S^3 branched over a knot. The 4-dimensional case has been considered by Piergallini and

---

2020 *Mathematics Subject Classification.* Primary 57M12; Secondary 57K40.

Key words and phrases. Branched covering, intersection lattice, intersection form.

Iori [20, 21], who showed that any closed connected oriented PL 4-manifold M can be represented by a simple d-fold branched covering $p\colon M \xrightarrow{d:1} S^4$ for any $d \geqslant 4$, where the branch set $B_p \subset S^4$ is a locally flat PL submanifold if $d \geqslant 5$ and has at most nodal singularities if $d = 4$.

In general, one would hope to give a complete answer to the following question.

Question. Given two PL n-manifolds M and N and a natural number $d \geqslant 2$, is there a d-fold branched covering

$$M \xrightarrow{d:1} N?$$

Criteria for the existence of a simple d-fold branched covering $p\colon M \xrightarrow{d:1} N$ with $N = \#_{k_1}\mathbb{C}\mathrm{P}^2 \#_{k_2} \overline{\mathbb{C}\mathrm{P}}^2$ or $N = \#_k(S^2 \times S^2)$ are given by Piergallini and Zuddas [22] in terms of the Betti numbers $b_2^\pm(M)$. Their proof heavily relies on the fact that the 4-manifold N is the union of a plumbing of disk bundles over 2-spheres with a single 4-handle. On the other hand, by a result of Neumann and Weintraub [19], the connected sums N as above are the only 4-manifolds which can be obtained by such a plumbing. In particular, the approach followed in [22] cannot be generalised to other choices of N.

In this paper, we develop new ideas to extend the above mentioned results of [22]. More precisely, we provide an answer to the above Question when N is a closed connected oriented PL 4-manifold admitting a handle decomposition with no 1- or 3-handles and $d \geqslant 4$.

Theorem A. *Let M and N be two closed connected oriented PL 4-manifolds such that N admits a handle decomposition with no 1- and 3-handles. For any $d \geqslant 4$, there exists a d-fold branched covering*

$$p\colon M \xrightarrow{d:1} N$$

if and only if there exists an isometric embedding of intersection lattices

$$d \cdot I_N \hookrightarrow I_M.$$

Moreover, if such branched covering exists, one can build it in such a way that it is simple and its branch set $B_p \subset N$ is a locally flat PL embedded surface if $d \geqslant 5$ while it has at most nodal singularities if $d = 4$.

In Proposition 3.2, we show that $d \cdot I_N$ can be isometrically embedded in I_M if and only if $b_2^\pm(N) \leqslant b_2^\pm(M)$ and we provide a list of achievable values of d, depending on the parity and the signature of the two intersection forms.

Remark. It is a long standing open problem in low dimensional topology that any closed PL simply connected 4-manifold N admits a handle decomposition with no 1- and 3-handles, see [15, Problem 4.18] in Kirby's list.

The next corollary follows from Theorem A combined with a theorem by Hopf [14], see also [4, Proposition 2.6]. In particular, it provides a criterion to check the existence of a dominant map $f\colon M \to N$ when N is a closed oriented smooth 4-manifold admitting a handle decomposition with no 1- and 3-handles, c.f. [4, Question 2.2].

Corollary B. *Let M and N be as in Theorem A. Then there exists a branched covering $p\colon M \to N$ if and only if there is a continuous dominant map $f\colon M \to N$ of positive degree.*

Remark. Corollary B provides a new proof of a particular case in [7, Theorem 3]. More preciely, Duan and Wang show that, given two closed connected oriented 4-manifolds M and N with N simply connected, there exists a dominant map $f\colon M \to N$ of degree d if and only if there is an isometric embedding of intersection lattices $d \cdot I_N \hookrightarrow I_M$. The two results coincide when either I_N is odd or I_N is even and $b_2(M) \geqslant 11/8\ |\sigma(M)|$, since in both cases N is homeomorphic to a 4-manifold with no 1-and 3-handles. This follows from Freedman's classification of simply connected topological 4-manifolds [9], Donaldson's diagonalization theorem of simply connected definite smooth 4-manifolds [5] and Serre's classification of integral unimodular bilinear forms [24, 17].

As an application of Theorem A and Proposition 3.2, we have the following result.

Corollary C. *Let M be a closed connected oriented PL 4-manifold. Then there exists a d-fold branched covering*

$$p\colon M \xrightarrow{d:1} \mathrm{K3}$$

for a suitable integer $d \geqslant 1$ if and only if $b_2^+(M) \geqslant 3$ and $b_2^-(M) \geqslant 19$. Moreover, if this is the case, the covering p can assumed to be simple and its degree can be any of the following: $d = 4, 6$ if β_M is odd; $d = 4, 8$ if β_M is even. Finally, the branch set B_p can be assumed to be a nodal surface if $d = 4$ and a locally flat embedded surface whenever $d \geqslant 5$.

The next corollary is an immediate consequence of Theorem A and Proposition 3.2. The proof is hence omitted.

Corollary D. *Let N be a closed connected oriented PL 4-manifold admitting a handle decomposition with no 1- and 3-handles and let N'*

be any smooth 4-manifold homeomorphic but not necessarily diffeomorphic to N. Then there exists a d-fold branched covering

$$p\colon N' \xrightarrow{d:1} N$$

for a suitable $d \geqslant 4$, according to Proposition 3.2. In particular, the value $d = 4$ is always allowed. Moreover, if such branched covering exists, one can build it in such a way that it is simple and its branch set $B_p \subset N$ is a locally flat PL surface if $d \geqslant 5$ while it has at most nodal singularities if $d = 4$.

Acknowledgments. Daniele Zuddas is a member of GNSAGA – Istituto Nazionale di Alta Matematica ‘Francesco Severi’, Italy.

Notation and conventions. We work in the category of PL manifolds. Notice that this coincides with the category of smooth manifolds for dimension $\leqslant 4$.

If M is an oriented manifold, we indicate by $\overline{M}$ the same manifold with reversed orientation.

Given a locally flat PL immersed submanifold $S \subset N$, we denote by $\nu(S) \subset N$ one of its closed tubular neighborhoods.

2. BRANCHED COVERINGS

We start this section by recalling the definition of branched covering in the PL category.

Definition 2.1. Given two compact PL n-manifolds M and N, a PL map

$$p\colon M \xrightarrow{d:1} N$$

is called a *d-fold branched covering* if it is non-degenerate and it restricts to a d-fold ordinary covering over the complement of a codimension 2 closed subpolyhedron of N. The *branch set* of p is the smallest subpolyhedron $B_p \subset N$ with such property, while the *degree* $d = d(p)$ of p is the cardinality of the preimage of any point in $N \setminus B_p$.

Moreover, p is called a *simple* branched covering, if the monodromy of any meridian of the branch set B_p is a transposition.

A d-fold branched covering as above is fully determined up to PL homeomorphisms by the pair (N, B_p) and the *monodromy* representation

$$\omega_p\colon \pi_1(N \setminus B_p) \longrightarrow S_d$$

associated to the ordinary covering $p|\colon M \setminus p^{-1}(B_p) \xrightarrow{d:1} N \setminus B_p$, where S_d denotes the permutation group of $\{1, \dots, d\}$ (see [8]).

Remark 2.2. The monodromy of a given branched covering p is usually described by means of its values of a given generator system for $\pi_1(N \setminus B_p)$. This is usually chosen to be a Hurwitz system when $N = S^2$ and B_p is a finite set of points. If $N = S^3$ and B_p is a link, we fix a projection diagram and consider a Wirtinger system of meridians. Attaching to any such generator of $\pi_1(S^n \setminus B_p)$ its image through the monodromy ω_p is called a *labeling* or a *coloring* of B_p.

3. Intersection forms

Given a smooth closed oriented 4-manifold M, we denote by

$$\beta_M \colon H_2(M;\mathbb{Z})/\operatorname{Tor} H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z})/\operatorname{Tor} H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

its intersection form on second homology modulo torsion and by

$$I_M := (H_2(M;\mathbb{Z})/\operatorname{Tor} H_2(M;\mathbb{Z}), \beta_M)$$

the integral unimodular inner product intersection space of M. If k is a non-zero integer, we set

$$k \cdot I_M := (H_2(M;\mathbb{Z})/\operatorname{Tor} H_2(M;\mathbb{Z}), k \cdot \beta_M).$$

In particular, we indicate by

$$\langle k \rangle = k \cdot \langle 1 \rangle$$

where $\langle 1 \rangle$ is the rank one positive definite unimodular lattice. Moreover, we let H and E_8 respectively denote the unimodular lattices represented by the homonymous matrices

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

We start by proving the following lemma, which immediately gives one of the two implications of Theorem A.

Lemma 3.1. *Let $p \colon M \xrightarrow{d:1} N$ be a d-fold branched covering between two closed connected oriented PL 4-manifolds. Then $d \cdot I_N$ admits an isometric embedding in I_M.*

Proof. Let $S_1, \ldots, S_k \subset N$ be PL embedded surfaces representing a basis of the free abelian group $H_2(N;\mathbb{Z})/\operatorname{Tor} H_2(N;\mathbb{Z}) \cong \mathbb{Z}^k$, where

$k = b_2(N)$. Without loss of generality, we can assume that all such surfaces are transverse to the branch set $B_p \subset N$ and we set $\widetilde{S}_i = p^{-1}(S_i)$ for $i = 1, \dots, k$. Then $\widetilde{S}_1, \dots, \widetilde{S}_k \subset M$ are PL embedded surfaces whose homology classes satisfy

$$\beta_M([\widetilde{S}_i], [\widetilde{S}_j]) = d \cdot \beta_N([S_i], [S_j])$$

for all $i, j = 1, \dots, k$. The conclusion follows. $\square$

The next proposition provides an easy-to-check criterion for understanding the thesis Lemma 3.1.

Proposition 3.2. *Let M and N be closed connected oriented PL 4-manifolds. Then there exists a positive integer d such that $d \cdot I_N$ admits an isometric embedding in I_M if and only if $b_2^+(N) \leqslant b_2^+(M)$ and $b_2^-(N) \leqslant b_2^-(M)$. Moreover, if these inequalities are satisfied, we can choose d as follows, for every $k \geqslant 1$.*

Case	d	β_N	β_M	*extra conditions*
1	1	*odd*	*odd*	
2	5			$b_2(N) \leqslant b_2(M)/2$
3	$2k$		*even*	$\sigma(M) = 0$
4	$2, 4, 6$			$\sigma(M) \neq 0$
5	$2k$	*even*	*odd*	$\sigma(N) = 0$
6	$2, 4, 6$			$\sigma(N) \neq 0$
7	k		*even*	$\sigma(N) = 0$
8	$4, 8, 12$			$\sigma(N) \neq 0$

In addition, if d is attainable, then so is $h^2 d$ for all integers $h \geqslant 1$.

For proving this, we need the following preliminary lemmas.

Lemma 3.3. *For every even $d \geqslant 2$, there is an isometric embedding*

$$\langle d \rangle \oplus \langle -d \rangle \hookrightarrow H.$$

Proof. Let e_1, e_2 be a basis of the hyperbolic lattice H such that $e_i^2 = 0$ and $e_1 \cdot e_2 = e_2 \cdot e_1 = 1$. For every $k \geqslant 1$, it is easy to verify that the vectors $e_1 + k e_2$ and $e_1 - k e_2$ span a sublattice of H isomorphic to $\langle 2k \rangle \oplus \langle -2k \rangle$, and let $2k = d$. $\square$

Lemma 3.4. *There are isometric embeddings*

$$d \cdot E_8 \hookrightarrow \oplus_8 \langle 1 \rangle$$

for every $d \in \{2, 4, 6\}$.

Proof. Consider the matrix

$$L_2 = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and observe that ${}^tL_2L_2 = 2E_8$, which implies that the sublattice of $\oplus_8\langle 1\rangle$ spanned by the columns of L_2 is isomorphic to $2 \cdot E_8$. Notice that $L_2 = 2G^{-1}$, where G is the matrix considered in [22, proof of Lemma 5.2].

Consider now the block matrices of order 8

$$G_2 = \oplus_4 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad G_3 = \oplus_2 \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

We notice that

$${}^tG_2G_2 = 2I_8 \quad \text{and} \quad {}^tG_3G_3 = 3I_8$$

where by I_n we denote the identity matrix of order n.

Now set

$$L_4 = G_2L_2 \quad \text{and} \quad L_6 = G_3L_2.$$

Then we have ${}^tL_dL_d = dE_8$ for $d = 4, 6$, and so L_d defines the desired embedding. □

Lemma 3.5. *There is an isometric embedding*

$$d \cdot E_8 \hookrightarrow \oplus_8 H$$

for $d \in \{4, 8, 12\}$.

Proof. There is a sequence of isometric embeddings

$$d \cdot E_8 \hookrightarrow \oplus_8\langle 2\rangle \hookrightarrow \oplus_8\langle 2\rangle \oplus_8 \langle -2\rangle \hookrightarrow \oplus_8 H$$

where the first and the last arrows follow from Lemma 3.4 and Lemma 3.3 respectively and the second arrow is the obvious one. □

Lemma 3.6. *There is an isometric embedding*

$$d \cdot E_8 \hookrightarrow E_8$$

for $d \in \{4, 8, 12\}$.

Proof. The proof of [22, Lemma 5.3] implies that there is an embedding

$$\oplus_8 \langle k \rangle \hookrightarrow E_8$$

for $k \in \{2, 4, 6\}$. One can then make the following composition of embeddings

$$2k \cdot E_8 \hookrightarrow \oplus_8 \langle k \rangle \hookrightarrow E_8$$

where the first arrow comes from Lemma 3.4, and let $2k = d$. □

Proof of Proposition 3.2. First of all we observe that the last part of the statement is immediate, as it is sufficient to multiply by h any given isometric embedding $\varphi \colon d \cdot I_N \hookrightarrow I_M$.

Suppose that the inequalities $b_2^\pm(M) \geqslant b_2^\pm(N)$ hold. We subsequently analyse all the possible cases as in the table in the statement.

Case 1: β_N and β_M odd. In this case I_N and I_M are diagonalizable by Donaldson's theorem [6] and the Serre classification of indefinite integral unimodular inner product spaces [24, 17]. The inequalities $b_2^\pm(M) \geqslant b_2^\pm(N)$ then trivially imply the existence of an embedding $I_N \hookrightarrow I_M$.

Case 2: β_N and β_M odd, $2b_2(N) \leqslant b_2(M)$. There is an isometric embedding

$$5 \cdot I_N \hookrightarrow I_M$$

by [22, Lemma 5.2].

Case 3: β_N odd, β_M even and $\sigma(M) = 0$. In this case, I_N is diagonalizable while $I_M \cong \oplus_{b_2^\pm(M)} H$ by the Serre classification. Lemma 3.3 implies the existence of an embedding

$$2k \cdot I_N \hookrightarrow I_M$$

for every $k \geqslant 1$.

Case 4: β_N odd, β_M even and $\sigma(M) \neq 0$. This case follows from an argument employed in [22, Lemma 5.2]. More precisely, there is an isomorphism

$$I_M \cong \oplus_{\frac{|\sigma(M)|}{8}} (\pm E_8) \oplus_{b_2^\mp(M)} H$$

by Donaldson's theorem and the Serre classification. In the proof of [22, Lemma 5.2], it is shown that there are isometric embeddings

$$\oplus_8 \langle d \rangle \hookrightarrow E_8$$

for $d \in \{2, 4, 6\}$. Since we have also isometric embeddings

$$\langle d \rangle \oplus \langle -d \rangle \hookrightarrow H$$

for the same values of d by Lemma 3.3, the conclusion follows.

Cases 5 and 7: β_N even and $\sigma(N) = 0$. These have been addressed in [22, Lemma 5.3].

Case 6: β_N even, β_M odd and $\sigma(N) \neq 0$. By Donaldson's theorem and the Serre classification, there is an isomorphism

$$I_N \cong \oplus_{\frac{|\sigma(N)|}{8}}(\pm E_8) \oplus_{b_2^{\mp}(N)} H. \tag{3.1}$$

Lemma 3.4 and [22, Lemma 5.3] imply the existence of isometric embeddings

$$d \cdot \left(\oplus_{\frac{|\sigma(N)|}{8}}(\pm E_8) \oplus_{b_2^{\mp}(N)} H\right) \hookrightarrow \oplus_{b_2^{+}(N)}\langle 1\rangle \oplus_{b_2^{-}(N)} \langle -1\rangle \hookrightarrow I_M$$

for every $d \in \{2, 4, 6\}$.

Case 8: β_N and β_M even and $\sigma(N) \neq 0$. In this case, we have that

$$I_M \cong \oplus_{\frac{|\sigma(M)|}{8}}(\pm E_8) \oplus_{b_2^{\mp}(M)} H. \tag{3.2}$$

Suppose first that $\sigma(N)\sigma(M) \leqslant 0$. By referring to equations 3.1 and 3.2, one can check that the inequality

$$b_2(N) + |\sigma(N)| \leqslant b_2(M) - |\sigma(M)|$$

holds. This means that the number of the hyperbolic direct summands plus eight times the number of the $\pm E_8$ direct summands of I_N is less or equal than the number of the hyperbolic direct summands of I_M, whence inside I_M there are sufficiently many hyperbolic direct summands in which we can embed $d \cdot I_N$ for $d = 4, 8, 12$, using Lemma 3.5 and [22, Lemma 5.3].

Assume now that $\sigma(N)\sigma(M) > 0$. By hypothesis, we have that

$$b_2^{\mp}(N) \leqslant b_2^{\mp}(M) \quad \text{and} \quad |\sigma(N)| + b_2^{\mp}(N) \leqslant |\sigma(M)| + b_2^{\mp}(M) \tag{3.3}$$

where we again refer to equation 3.1 and equation 3.2 for I_N and I_M respectively. Since $b_2^{\pm}(N) \leqslant b_2^{\mp}(M)$, we can embed d times the hyperbolic direct summands of I_N into the hyperbolic direct summands of I_M, using case 7. If $|\sigma(N)| \leqslant |\sigma(M)|$, we then use Lemma 3.6 to embed d times the $\pm E_8$ direct summands of I_N into the $\pm E_8$ direct summands of I_M. If $|\sigma(N)| > |\sigma(M)|$, we observe that the second inequality in 3.3 implies that

$$|\sigma(N)| - |\sigma(M)| \leqslant b_2^{\mp}(M) - b_2^{\mp}(N),$$

which means that eight times the number of remaining $\pm E_8$ direct summands in I_N is less or equal than the number of remaining H direct summands in I_M. We can hence embed d times such $\pm E_8$ direct summands into the remaining H direct summands of I_M, using Lemma 3.5.

For the converse, suppose that there is an isometric embedding of $d \cdot I_N$ inside I_M for some integer number d. Then the equalities $b_2^\pm(M) \geqslant b_2^\pm(N)$ follow by diagonalizing $d \cdot I_N \otimes_{\mathbb{Z}} \mathbb{R}$ and $I_M \otimes_{\mathbb{Z}} \mathbb{R}$ and by the definition of $b_2^\pm$. ☐

4. Fillability of 3-dimensional branched coverings

Definition 4.1. The cobordism group of simple d-fold branched coverings over S^3 is defined as

$$\Gamma_d(S^3) = \{p\colon Y \xrightarrow{d:1} S^3 \text{ simple } \mid B_p \subset S^3 \text{ a link}\}/{\sim},$$

where $p_1 \sim p_2$ if and only if there exists a simple d-fold branched covering

$$c : C \xrightarrow{d:1} S^3 \times [0,1]$$

which is a cobordism between p_1 and p_2. Moreover, we require that the branch set $B_c \subset S^3 \times [0,1]$ is a proper locally flat PL surface.

We endow $\Gamma_d(S^3)$ with a binary operation by setting $[p_1] + [p_2] = [p]$, where p is the simple d-fold covering over S^3 whose labeled branch set B_p is given by the disjoint separate union of B_{p_1} and B_{p_2} inside S^3, for all $[p_1], [p_2] \in \Gamma_d(S^3)$. This makes $\Gamma_d(S^3)$ an abelian group.

The following result will be our main tool to extend a given simple branched covering over a 4-handle.

Theorem 4.2. *Let W^4 be a compact connected PL 4-manifold with non-empty boundary, and let*

$$p\colon \partial W^4 \xrightarrow{d:1} S^3$$

be a simple d-fold covering branched over a link $B_p \subset S^3$ such that $[p] = 0 \in \Gamma_d(S^3)$, where $d \geqslant 4$. Then p extends to a simple d-fold covering

$$q\colon W^4 \xrightarrow{d:1} B^4$$

branched over a PL embedded surface $B_q \subset B^4$ which is locally flat if $d \geqslant 5$, while it has at most nodal singularities if $d = 4$.

Proof. It is a direct consequence of [23, Theorem 1.4], in light of [2, Lemma 1.14]. More precisely, the condition $[p] = 0 \in \Gamma_d(S^3)$ is equivalent to asking that the branch set $B_p \subset S^3$ bounds a locally flat proper surface in B^4 over which the monodromy representation of p extends. But then one can choose such surface to be ribbon because of [2, Lemma 1.14] and the conclusion is a direct application of [23, Theorem 1.4]. ☐

5. Proof of Theorem A

This section is devoted to the proof of Theorem A. Before that, we need to state and demonstrate the following lemma.

Lemma 5.1. *Let* $A = A_1 \sqcup \cdots \sqcup A_n$ *and* $B = B_1 \sqcup \cdots \sqcup B_n$ *be two oriented links in* S^3 *such that* $\operatorname{lk}(A_i, A_j) = \operatorname{lk}(B_i, B_j)$ *for every* $i \neq j$*. Then, for every* $i = 1, \ldots, n$, A_i *is related to* B_i *via a finite sequence of oriented band attachments and ambient isotopy. Moreover, each band can be chosen to intersect the link only at its attaching edges, both lying in the same connected component of the link.*

Proof. For every $i = 1, \ldots, n$, let $F_i \subset S^3 \times [0, 1]$ be a PL locally flat proper oriented surface such that $\partial F_i = A_i \sqcup \overline{B}_i$, where we view $A \subset S^3 \times \{0\}$ and $B \subset S^3 \times \{1\}$ with $S^3 \times \{0\}$ and $S^3 \times \{1\}$ oriented as the boundary of $S^3 \times [0, 1]$. Without loss of generality, we can assume that each F_i is connected and that $F = F_1 \cup \cdots \cup F_n \subset S^3 \times [0, 1]$ is self-transversely immersed. Now, notice that the algebraic intersection between F_i and F_j, for $i \neq j$, is given by

$$F_i \cdot F_j = \operatorname{lk}(A_i, A_j) - \operatorname{lk}(\overline{B}_i, \overline{B}_j) = 0,$$

where the minus sign arises from the fact that $S^3 \times \{0\}$ and $S^3 \times \{1\}$ are endowed with opposite orientations. Since $F_i \cdot F_j$ vanishes, we can geometrically remove all the intersection points by tubing them in pairs with opposite signs. We can hence reduce to the case in which F is nonsingular. Without loss of generality, we can assume that the restriction to F_i of the height function $S^3 \times [0, 1] \to [0, 1]$ is Morse. If all the critical points of this map are saddle points, then the conclusion follows. Suppose now that such map has a local minimum in $\operatorname{Int} F_i$. We can then push this minimum to the lower boundary component $S^3 \times \{0\}$ along an arc that does not intersect F elsewhere. This procedure introduces a new trivial component in the link A, which can be trivially merged to A_i with a suitable oriented band in $S^3 \times \{0\}$. By pushing the interior of such band into $\operatorname{Int} S^3 \times [0, 1]$, we get a link in $S^3 \times \{0\}$ which is ambiently isotopic to A. We repeat this procedure for all the local minima. In a similar way, we can remove the local maxima, and the conclusion follows. □

Proof of Theorem A. If $p \colon M \xrightarrow{d:1} N$ is a simple branched covering, then Lemma 3.1 implies the existence of an isometric embedding of intersection lattices $d \cdot I_N \hookrightarrow I_M$.

Let us now prove the converse. Let

$$L = L_1 \cup \cdots \cup L_n \subset S^3$$

be an oriented framed link representing the 2-handlebody N_2 of N. Then L induces a basis $\alpha_1, \dots, \alpha_n$ of $H_2(N; \mathbb{Z})$, where each α_i is the homology class of a closed connected oriented surface in N obtained by taking the union of a Seifert surface of L_i pushed inside B^4 and of the core of the 2-handle attached along L_i. Let $A = (a_{ij} = \beta_N(\alpha_i, \alpha_j)) \in \mathrm{GL}(n, \mathbb{Z})$ be the matrix representing the intersection lattice I_N of N with respect to the basis $\alpha_1, \dots, \alpha_n$, so that a_{ii} is the framing of L_i for $i = 1, \dots, n$ and $a_{ij} = \mathrm{lk}(L_i, L_j)$ for all $i \neq j$.

Suppose now that there is an isometric embedding of intersection lattices

$$\varphi\colon d \cdot I_N \hookrightarrow I_M.$$

Let $S_i \subset M$ be a closed connected oriented locally flat PL surface representing $\varphi(\alpha_i)$ for $i = 1, \dots, n$. In particular, we have that

$$\beta_M([S_i], [S_j]) = \beta_M(\varphi(\alpha_i), \varphi(\alpha_j)) = d \cdot \beta_N(\alpha_i, \alpha_j) = d \cdot a_{ij}$$

for every $i, j \in \{1, \dots, n\}$. Up to possibly a small perturbation, we can assume that the surfaces S_i are in general position, so that they are transversal to each other and no three of them have non-empty intersection. Up to tubing, we can also assume that the geometric intersection of S_i and S_j equals the algebraic intersection for all $i \neq j$. A regular neighborhood of

$$S = S_1 \cup \cdots \cup S_n$$

is then a plumbing of normal disk bundles over $S_1, \dots, S_n$ with intersection matrix $d \cdot A$. At this point, pick a small 4-ball around each double point of S. If there is a surface S_i such that $S_i \cap S_j = \emptyset$ for every $j \neq i$, pick a 4-ball that intersects S_i along a properly embedded trivial 2-disk. Now join all these balls by suitable 1-handles embedded in the complement of a regular neigbourhood of S to get a 4-ball $B^4 \subset M$ whose boundary S^3 is transversal to all the surfaces S_i and let $\widetilde{L}_i = S_i \cap S^3$ for $i = 1, \dots, n$. In this way we get an oriented link

$$\widetilde{L} := \widetilde{L}_1 \sqcup \cdots \sqcup \widetilde{L}_n \subset S^3.$$

where $\widetilde{L}_i$ is oriented as the boundary of $S_i \cap B^4$.

Now notice that, if $\beta \subset S^3$ is an oriented band attached to $\widetilde{L}_i$ in the complement of $\cup_{j \neq i} \widetilde{L}_j$ which realizes a homology of $\widetilde{L}_i$ inside $S^3 - \cup_{j \neq i} \widetilde{L}_j$, we can push β inside and outside B^4 to get bands β' and β'' respectively. Their union $\beta' \cup \beta''$ is then an oriented tube that realizes a homology of S_i in M. By Lemma 5.1, we can hence modify each S_i up to oriented homology so that the intersection of the plumbing S with $S^3 = \partial B^4$ is any oriented link with linking matrix $d \cdot A$. Notice that these tubing operations do not change the intersection numbers

$\beta_M([S_i],[S_j]) = d \cdot a_{ij}$ for every i, j, since they change the surfaces without affecting their homology classes.

We are now ready to construct the desired branched covering between M and N. The starting point is the d-fold simple covering

$$q_0 \colon B^4 \xrightarrow{d:1} B^4$$

branched over $d-1$ pairwise separated trivial properly embedded 2-disks with labeling $(1\ 2), \dots, (d{-}1\ d)$ respectively. Up to a suitable isotopy of the branch set B_{q_0}, we can assume that L is disjoint from B_{q_0} and each component L_i of L and has a cycle of length d as monodromy, so that its preimage $q_0^{-1}(L_i) \subset S^3$ is connected, see Figure 1.

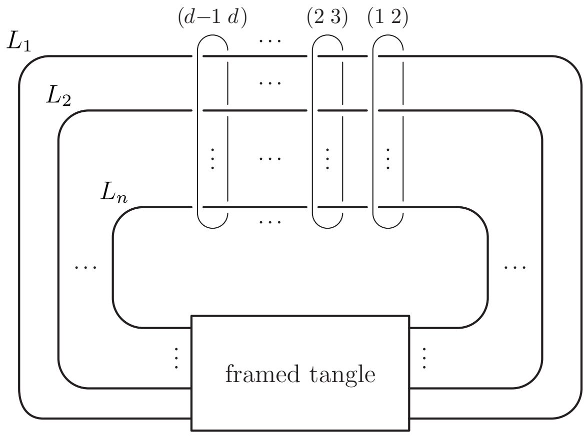

FIGURE 1. Labeled branch set of $q_0|_\partial$.

Now notice that, by construction, the intersection matrix of the link

$$q_0^{-1}(L_1) \sqcup \cdots \sqcup q_0^{-1}(L_n)$$

is equal to $d \cdot A$ and this implies that each component $q_0^{-1}(L_i)$ is realizable as the intersection $\widetilde{L}_i$ of S_i with S^3 as above.

We are now going to extend q_0 over the 2-handles of N. For every $i = 1, \dots, n$,

$$\Sigma_i := S_i \setminus \operatorname{Int} B^4$$

is a surface with one boundary component $\widetilde{L}_i = S_i \cap S^3$ with a d-fold covering

$$p_i \colon \Sigma_i \xrightarrow{d:1} B^2$$

branched over $2g(\Sigma_i) + d - 1$ points, the meridians of $2g(\Sigma_i) + 1$ of them have monodromy $(1\ 2)$ and there is exactly one branch point corresponding to each transposition $(2\ 3), \dots, (d{-}1\ d)$. Here $g(\Sigma_i)$ denotes the genus of the surface Σ_i. We remark that such p_i can be constructed by stabilizing the 2-fold branched covering $\Sigma_i \xrightarrow{2:1} B^2$ obtained by removing two equivariant disks from the domain and target of the quotient map $\Sigma_i \cup_\partial B^2 \xrightarrow{2:1} S^2$ of the hyperelliptic involution on the genus $g(\Sigma_i)$ closed oriented surface $\Sigma_i \cup_\partial B^2$. Now notice that $\nu(S_i) \setminus \mathrm{Int}\, B^4 \cong \Sigma_i \times B^2$ meets B^4 along $\nu(\widetilde{L}_i) \subset S^3 = \partial B^4$. In particular, $\partial\Sigma_i \times B^2$ is identified with $\nu(\widetilde{L}_i)$ with framing $\beta_M([S_i],[S_i]) = d \cdot a_{ii}$. Since this coincides with the pull-back of the framing a_{ii} of L_i under q_0, we can extend $q_0 \colon B^4 \xrightarrow{d:1} B^4$ over the 2-handles of N by the simple branched coverings

$$\nu(S_i) \setminus \mathrm{Int}\, B^4 \cong \Sigma_i \times B^2 \xrightarrow{d:1} B^2 \times B^2$$

given by $p_i \times \mathrm{id}_{B^2}$. In particular, extending q_0 over the 2-handle attached along L_i adds to the branch set $2g(\Sigma_i) + d - 1$ embedded 2-disks given by parallel copies of the co-core, where $2g(\Sigma_i) + 1$ of them have monodromy $(1\ 2)$ and the other 2-disks are labeled by $(2\ 3), \dots, (d{-}1\ d)$ respectively. In this way, we have just found a simple d-fold branched covering

$$q_2 \colon B^4 \cup \nu(S) \xrightarrow{d:1} N_2$$

where N_2 denotes the 2-handlebody of N, as above.

In order to conclude, we need to further extend q_2 over the unique 4-handle of N. We will show that the restriction of q_2 to the boundary is trivial in the cobordism group $\Gamma_d(S^3)$, see Definition 4.1. The conclusion will then follow from Theorem 4.2.

One can visualize the branch set of $q_2|_\partial \colon \partial\nu(S) \xrightarrow{d:1} S^3$ as follows. First of all, recall that we isotoped the link L as in Figure 1, so that the meridian of each component L_i has a permutation of maximum length as a monodromy. Notice that, since N has no 3-handles, S^3 is diffeomorphic to the boundary of the 2-handlebody of N and the framed link L provides a Dehn surgery diagram for S^3. In this surgery diagram, one can visualize the dual 2-handles of N as 0-framed meridians of the components of L, see [10, Example 5.5.5]. In particular, in the surgery presentation of S^3 given by the framed link L, the labeled branch set of $q_2|_\partial$ is the union of $d-1$ components as in Figure 1 and $2g(\Sigma_i) + d - 1$ parallel meridians of each L_i, $2g(\Sigma_i) + 1$ of them with monodromy $(1\ 2)$ and exactly one of them for each monodromy of type $(2\ 3), \dots, (d{-}1\ d)$, see Figure 2.

We will now construct a surface bounding $B_{q_2|_\partial}$, over which the monodromy $\omega_{q_2|_\partial}$ extends. This will imply that $[q_2|_\partial] = 0 \in \Gamma_d(S^3)$ and the conclusion will follow from Theorem 4.2. For every $i = 1, \dots, n$, the first $2g(\Sigma_i)$ meridians of L_i with monodromy $(1\ 2)$ bound $g(\Sigma_i)$ disjointly embedded labeled cylinders. The union of the remaining

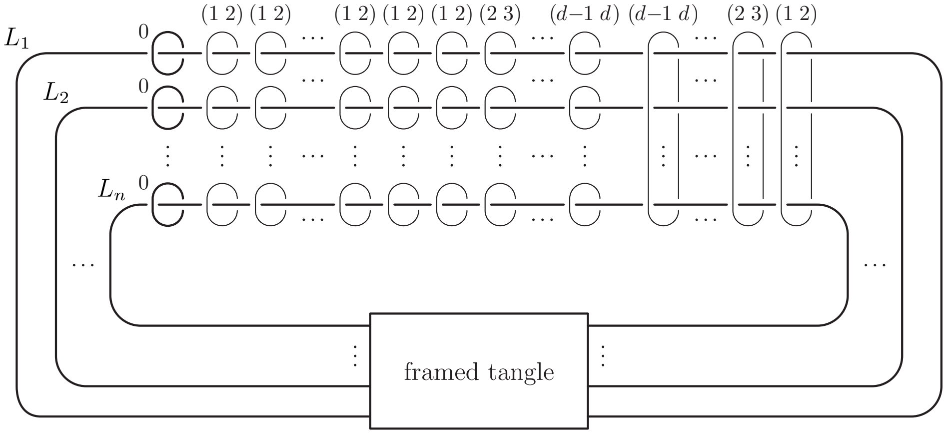

FIGURE 2. Labeled branch set of $q_2|_\partial$ in the surgery diagram of S^3 given by the framed link L.

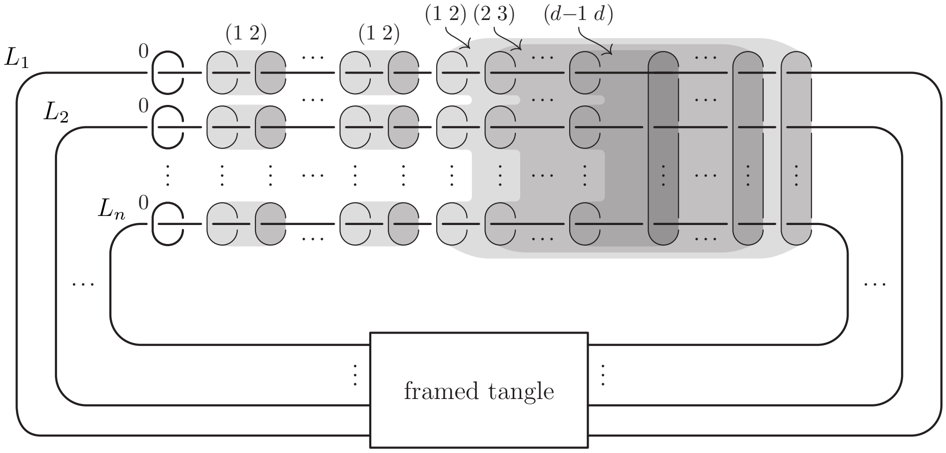

FIGURE 3. Labeled surface bounding $B_{q_2|_\partial}$.

(1 2)-meridians of L with the (1 2)-component of $B_{q_2|_\partial}$ coming from B_{q_0} bounds an embedded colored surface, see Figure 3. In particular, such surface is an embedded S^2 with $n+1$ punctures. Similarly, for $i = 2, \dots, d-1$, there are disjointly embedded genus 0 labeled surfaces with $n+1$ boundary components, co-bounding the $(i\ \ i{+}1)$-framed meridians of L and the $(i\ \ i{+}1)$-framed component coming from the branch set of q_0. The conclusion follows. □

6. PROOFS OF COROLLARIES B AND C

Proof of Corollary B. One implication is trivial, since any branched covering $p\colon M \to N$ is a dominant map. For the converse, assume that $f\colon M \to N$ is dominant and that $\deg(f) \geqslant 1$. By [14] (see also [4, Proposition 2.6]), the pull-back map

$$f^*\colon H^*(N;\mathbb{Q}) \longrightarrow H^*(M;\mathbb{Q})$$

is an injective morphism of $\mathbb{Q}$-algebras. This implies the existence of an isometric embedding $\deg(f)\cdot I_N \hookrightarrow I_M$. The conclusion then follows from Theorem A. ☐

Proof of Corollary C. The K3-surface satisfies the hypothesis of Theorem A, since it admits a handle decomposition without 1- and 3-handles (see [10, Figure 8.15] for $n = 2$ or [11, Figure 2.15]). Therefore, there exists a d-fold branched covering $M \xrightarrow{d:1} \mathrm{K3}$ if and only if there is an isometric embedding of intersection lattices $d \cdot I_{\mathrm{K3}} \hookrightarrow I_M$. Given that $b_2^+(\mathrm{K3}) = 3$ and $b_2^-(\mathrm{K3}) = 19$, the conclusion follows from Proposition 3.2. ☐

References

[1] J. W. Alexander, *Note on Riemann spaces*, Bull. Amer. Math. Soc. **26** (1920), no. 8, 370–372.

[2] V. Bais, R. Piergallini, and D. Zuddas. *Branched covering representation of non-orientable 4-manifolds.* arXiv preprint arXiv:2509.09319 (2025).

[3] I. Berstein and A. L. Edmonds, *On the construction of branched coverings of low-dimensional manifolds*, Trans. Amer. Math. Soc. **247** (1979), 87–124.

[4] P. de la Harpe, *Brouwer degree, domination of manifolds, and groups presentable by products*, Bulletin of the Manifold Atlas (2017), 1–25, `http://www.boma.mpim-bonn.mpg.de/data/58screen.pdf`.

[5] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*. J. Diff. Geom. **18** (1983), no. 2, 279–315.

[6] S. K. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Diff. Geom. **26** (1987), 397–428.

[7] H. B. Duan and S. C. Wang, *Non-zero degree maps between 2n-manifolds*, Acta Math. Sin. **20** (2004), no. 1, 1–14.

[8] R. H. Fox, *Covering spaces with singularities*, Princeton Math. Ser. 12 (1957), 243–257.

[9] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982), no. 3, 357–453.

[10] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20, Amer. Math. Soc., Providence, RI, 1999.

[11] J. Harer, A. Kas and R. Kirby, *Handlebody decompositions of complex surfaces*, Mem. Amer. Math. Soc. **62** (1986), no. 350.

[12] H. M. Hilden, *Every closed orientable 3-manifold is a 3-fold branched covering space of S^3*, Bull. Amer. Math. Soc. **80** (1974), 1243–1244.

[13] U. Hirsch, *Über offene Abbildungen auf die 3-Sphäre*, Math Z. **140** (1974), 203–230.

[14] H. Hopf, *Zur Algebra der Abbildungen von Mannigfaltigkeiten*, J. Reine Angew. Math. **163** (1930), 71–88.

[15] R. C. Kirby, *Problems in Low-Dimensional Topology*, AMS/IP Studies in Advanced Mathematics, Geometric topology (Athens, GA, 1993), 35–473, American Mathematical Society, Providence, RI, 1997.

[16] R. Mandelbaum, *Decomposing analytic surfaces*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), pp. 147–217, Academic Press, 1979.

[17] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete **73**, Springer-Verlag 1973.

[18] J. M. Montesinos-Amilibia, *A representation of closed orientable* 3*-manifolds as* 3*-fold branched coverings of* S^3, Bull. Amer. Math. Soc. **80** (1974).

[19] W. D. Neumann and S. H. Weintraub, *Four-manifolds constructed via plumbing*, Math. Ann. **238** (1978), no. 1, 71–78.

[20] R. Piergallini, *Four-manifolds as 4–fold branched covers of* S^4, Topology **34** (1995) 497–508.

[21] R. Piergallini and M. Iori, *4-manifolds as covers of the 4-sphere branched over non-singular surfaces*, Geometry & Topology **6** (2002), 393–401.

[22] R. Piergallini and D. Zuddas, *Branched coverings of* $\mathbb{C}\mathrm{P}^2$ *and other basic 4-manifolds*, Bull. Lond. Math. Soc. **53** (2021), no. 3, 825–842.

[23] R. Piergallini and D. Zuddas, *On branched covering representation of 4-manifolds*, J. Lond. Math. Soc. (2) **100** (2019), no. 1, 1–16.

[24] J. P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics **7**, Springer-Verlag 1973.

Scuola Internazionale Superiore di Studi Avanzati (SISSA), Via Bonomea 265, 34136, Trieste, Italy.
Email address: `vbais@sissa.it`

Scuola di Scienze e Tecnologie, Università di Camerino, Via Madonna delle Carceri 9, 62032 Camerino, Italy.
Email address: `riccardo.piergallini@unicam.it`

Dipartimento di Matematica, Informatica e Geoscienze, Università degli Studi di Trieste, Via Valerio 12/1, 34127, Trieste, Italy.
Email address: `dzuddas@units.it`