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REPRESENTING LINKS IN 3-MANIFOLDS BY BRANCHED COVERINGS OF S^3

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ABSTRACT – We introduce a planar coloured-diagram representation of links in 3-manifolds given as branched coverings of the 3-sphere. We also prove an equivalence theorem based on local moves and the existence of a universal configuration for such representation. As an application we give unified proofs of two different results on existence of fibered links in 3-manifolds.

Introduction and notations.

Throughout this paper we shall work in the piecewise linear (PL) category. We refer to [11] and [12] for basic notions and standard results.

An n-link in a 3-manifold M is a subspace $L \subset M$ homeomorphic to $L_n = S^1 \sqcup \ldots \sqcup S^1$ (the disjoint union of n copies of S^1). Two n-links $L, L' \subset M$ are said to be isotopically equivalent if there exists an ambient isotopy $H : M \times [0,1] \to M$ such that $h_1(L) = L'$. By [8], this notion of equivalence can be also formulated in terms of locally unknotted isotopy of L_n in M, or in terms of finite sequence of elementary deformations. We say that L' is obtained from L by an elementary deformation if there exists a disk $D \subset M$ such that $\alpha = L \cap D$ and $\alpha' = L' \cap D$ are arcs in $\operatorname{Bd} D$, $\alpha \cap \alpha' = \operatorname{Bd} \alpha = \operatorname{Bd} \alpha'$ and $L' = (L - \alpha) \cup \alpha'$.

If $p : M \to S^3$ is a d-fold covering branched over a link, then S_p, $B_p = p(S_p)$, $S'_p = p^{-1}(B_p) - S_p$ and $\omega_p : \pi_1(S^3 - B_p) \to \Sigma_d$ will denote respectively the singular link, the branching link, the pseudo-singular link and the monodromy of p. We recall that M is uniquely determined (up to homeomorphism) by B_p and ω_p.

By considering the identification $S^3 \cong R^3 \cup \{\infty\}$, the covering p can be represented by a planar diagram of B_p, whose overpasses are labelled with the monodromy of the corresponding Wirtinger meridians. Such a diagram will be called a *p-diagram* of M. Two different p-diagrams can be related by a finite sequence of labelled Reidemeister moves. Note that any p-diagram naturally induces a splitting complex for p, namely the half-cylinder under B_p.

Any closed orientable 3-manifold M admits a simple covering $p : M \to S^3$ branched over a knot, actually a 3-fold one ([4], [7] and [9]). Therefore, it can be represented by a p-diagram labelled by transpositions. The set of all simple coverings $p : M \to S^3$ branched over a link will be denoted by $\mathcal{S}(M)$.

Diagrams of links.

If M is a 3-manifold and $p : M \to S^3$ is a branched covering, then a link $L \subset M$ is called *p-regular* if $L \cap (S_p \cup S'_p) = \emptyset$ and $p_{|L}$ is injective. Standard general position arguments ensure that every link is ε-equivalent to a p-regular one. If L is a p-regular link, then the projection $p(L)$ is a link embedded in $S^3 - B_p$ and L is uniquely determined by $p(L)$, together with a choice of a point P_i of $p^{-1}(K_i) \cap L$, for each component K_i of $p(L)$. In fact, the link L can be reconstructed by lifting each K_i, starting from P_i.

Given a p-diagram $\mathcal{D}$ of a 3-manifold M, we obtain a planar representation of any link $L \subset M$ (up to isotopical equivalence), called a *link p-diagram*, in the following way. By general position, we can assume that L is p-regular and that the projection of $p(L)$ in the plane of $\mathcal{D}$ gives rise, together with $\mathcal{D}$ itself, to a diagram of $p(L) \cup B_p$, then we label each overpass C of $p(L)$ with the number of the sheet containing $p^{-1}(C) \cap L$ (sheets and their numbering come from the splitting complex and the monodromy associated to $\mathcal{D}$). Observe that consecutive overpasses of $p(L)$ are labelled by i and $\sigma(i)$ if the transition occurs under an overpass of B_p with label σ, while they have the same label if the transition occurs under an overpass of $p(L)$ itself. Figure 1 shows a diagram representing a knot in $S^2 \times S^1$. Here, as well as in all the figures representing link p-diagrams, thicker (resp. thinner) lines are used for the link $p(L)$ (resp. the branching set B_p).

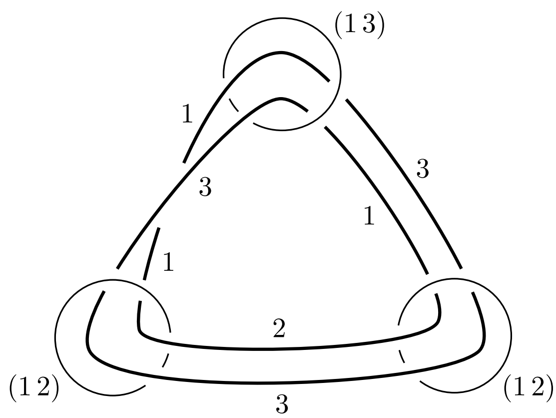

Figure 1.

By a *labelled Reidemeister move* on a link p-diagram we mean an ordinary Reidemeister move on the diagram, taking place inside a disk D, with the final stage labelled as the initial one outside D and in the unique possible way, according to continuity, inside D.

Since any ambient isotopy of S^3 rel B_p (uniquely) lifts to an ambient isotopy of M rel $S_p \cup S'_p$, any two link p-diagrams related by labelled Reidemeister moves which keep $\mathcal{D}$ fixed represent links isotopically equivalent in M.

On the other hand, in order to represent any isotopical equivalence of links in M we need the local moves I, II and III depicted in the following Fig. 2.

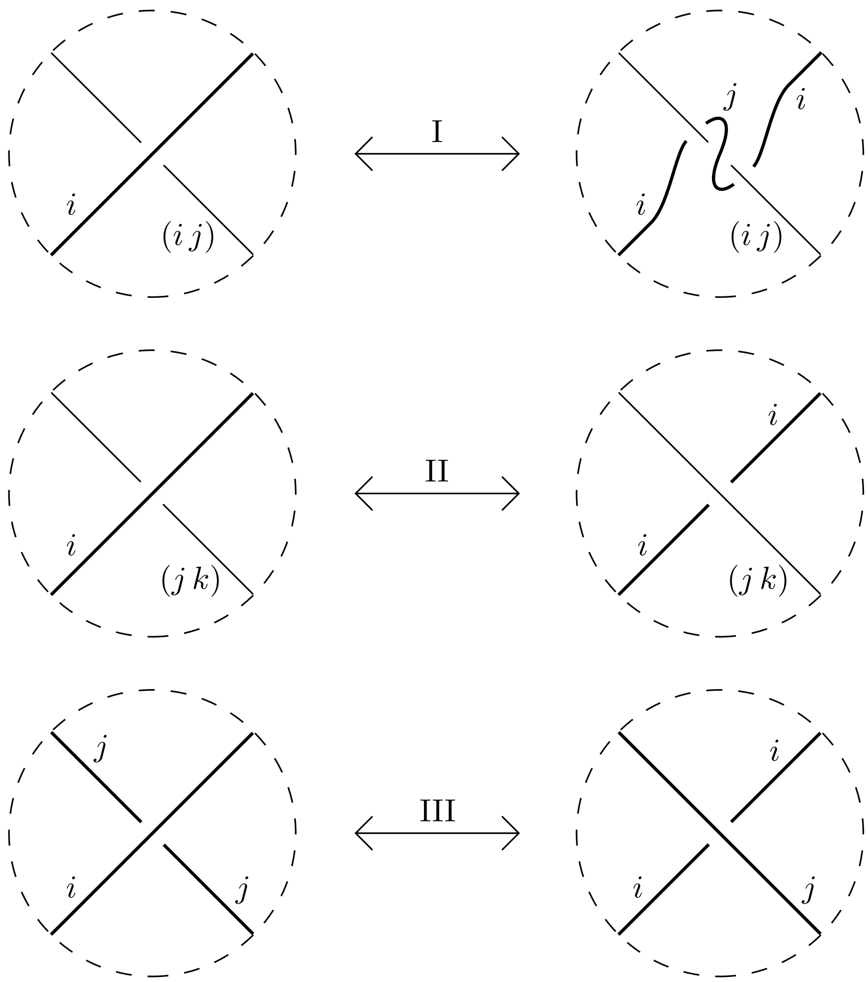

Figure 2.

Equivalence Theorem. *Let M be a closed orientable 3-manifold and $p \in \mathcal{S}(M)$. Then two p-diagrams of links based on the same p-diagram $\mathcal{D}$ of M, represent isotopically equivalent links in M if and only if they are related by a finite sequence of labelled Reidemeister moves fixing $\mathcal{D}$ and moves I, II and III.*

Proof. It can be easily realized that the moves I, II and III do not change the equivalence class of the link. We prove the "only if" part. Let $L, L' \subset M$ be isotopically equivalent p-regular links, there exists a locally unknotted isotopy $H : L_n \times [0,1] \to M$ such that $h_0(L_n) = L$ and $h_1(L_n) = L'$. We will denote by L_t the link $h_t(L_n)$, for each $t \in [0,1]$.

By general position, we can assume that the set $\operatorname{Sing} H$ of the singular points of $H(L_n \times [0,1])$ is a 1-subcomplex of M and that $H(L_n \times [0,1])$ meets transversally the singular set S_p in a finite number of non-singular points $P_i = H(x_i, t_i)$ with $i = 1, 2, \dots, n$, such that $0 < t_1 < t_2 < \cdots < t_n < 1$. For each $i = 1, 2, \dots, n$, let D_i be a closed disk in $H(L_n \times]t_{i-1}, t_{i+1}[) - (\operatorname{Sing} H \cup S'_p)$ such that $P_i \in \operatorname{Int} D_i$ and the set $L_{t_i} \cap \operatorname{Bd} D_i$ consists of exactly two points Q_i, R_i (put $t_0 = 0$ and $t_{n+1} = 1$). Moreover, let $\beta'_i = \{H(x,t) \in \operatorname{Bd} D_i \mid t < t_i\}$ and $\beta''_i = \{H(x,t) \in \operatorname{Bd} D_i \mid t > t_i\}$ be the components of $\operatorname{Bd} D_i - \{Q_i, R_i\}$. Up to ε-isotopy rel S_p, we can assume D_i as it appears in Fig. 3, where p is locally induced by the symmetry with respect to the vertical axis S_p.

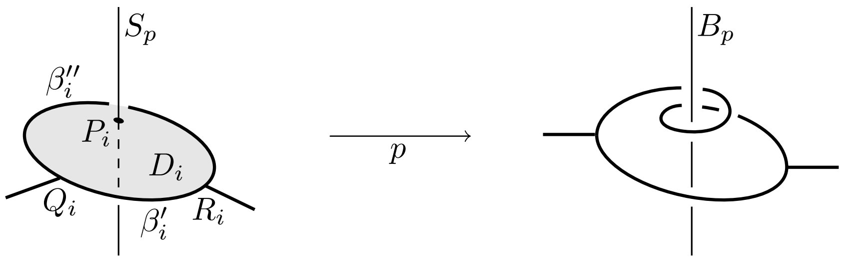

Figure 3.

Then, the links $L'_{t_i} = (L_{t_i} - \operatorname{Int} D_i) \cup \beta'_i$ and $L''_{t_i} = (L_{t_i} - \operatorname{Int} D_i) \cup \beta''_i$ are p-regular and their projections $p(L'_{t_i})$ and $p(L''_{t_i})$ differ by a local move I.

For each $i = 0, 1, \dots, n$, the links L''_{t_i} and $L'_{t_{i+1}}$ are isotopically equivalent in $M - S_p$ (put $L''_{t_0} = L$ and $L'_{t_{n+1}} = L'$). Then, there exists a finite sequence of links $L''_{t_i} = C_0, C_1, \dots, C_{n_i} = L'_{t_{i+1}}$ such that C_j is obtained from C_{j-1} by an elementary deformation in $M - S_p$ for each $j = 1, \dots, n_i$.

Moreover, we can assume that the disk D_j which realizes the elementary deformation between C_{j-1} and C_j is contained in a 3-ball $T_j \subset M - S_p$ such that $p_j = p_{|T_j} : T_j \to p(T_j)$ is a homeomorphism. Finally, by general position we can assume that D_j meets transversally S'_p and $p_j^{-1}(p(C_{j-1} - T_j) \cap p(T_j))$ in a finite number of interior points. Then, the projections $p(C_{j-1})$ and $p(C_j)$ differ (up to isotopy of S^3 rel B_p) by a finite sequence of local moves II and III. Namely, we have a move of type II for each point in $D_j \cap S'_p$ and a move of type III for each point in $D_j \cap p_j^{-1}(p(C_{j-1} - T_j) \cap p(T_j))$. $\square$

Remark 1. The theory could be extended to arbitrary (non necessarily simple) branched coverings $p : M \to S^3$. In this context we have, in addition to move III, a countable set of moves I_n (see Fig. 4). The first two of these moves, namely I_0 and I_1, coincide respectively with the moves II and I.

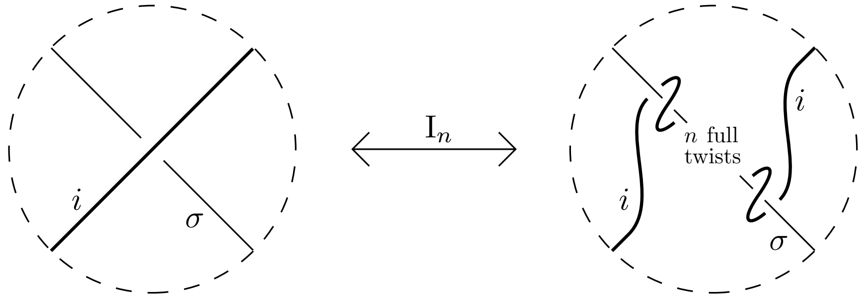

Figure 4.

Remark 2. In the next sections, we need to consider also the weaker notion of topological equivalence for links in a 3-manifold M, namely the equivalence up to homeomorphism (not necessarily isotopic to the identity). In this case all the labelled Reidemeister moves on diagrams (also the ones moving the branching set) can be allowed. Moreover, for simple 3-fold coverings diagrams, covering moves introduced in [10] could be used in order to make the representation independent from the covering.

Special configurations.

In this section we use the moves introduced above, in order to get special configurations for the projections of links. We start up by observing that projections of links can be untied.

Proposition 3. *Let M be a closed orientable 3-manifold and $p \in \mathcal{S}(M)$. Then every link $L \subset M$ is isotopically equivalent to a p-regular link $L' \subset M$ such that $p(L')$ is a trivial link in S^3.*

Proof. We can assume (up to ε-isotopy) that L itself is p-regular. We can also assume (up to isotopy) that L has a p-diagram such that any crossing of $p(L)$ involves strings

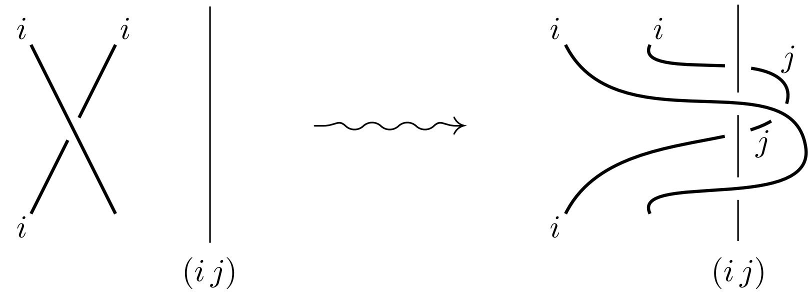

Figure 5.

with different labels (see Fig. 5). Then, we can invert a suitable set of crossings of $p(L)$, by using move III, in order to get a p-diagram of a link L' isotopically equivalent to L, such that $p(L')$ is a trivial link. $\square$

In the following, we need the move J depicted in Fig. 6, which modify the branching set keeping the link projection fixed. Such a move is equivalent (up to isotopy of S^3) to one move I or two moves II depending on the labels, so it changes a p-diagram of a link into a p'-diagram of a new link topologically equivalent to the original one (see remark 2). Of course, this last diagram can also be regarded as a p''-diagram of the original link itself. We remark that both the coverings p' and p'' are topologically equivalent to p.

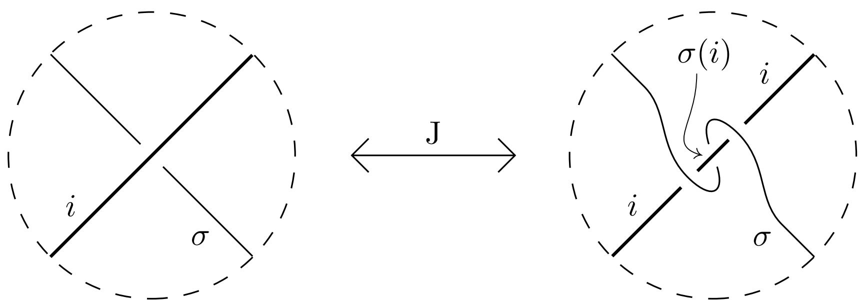

Figure 6.

Now, denoting by ρ, θ and z the usual cylindrical coordinates on R^3, let T_n be a trivial n-link whose k-th component is contained in the half-plane $\theta = 2\pi k/n$ and meets the truncated cylinder $C = \{(\rho, \theta, z) \mid \rho \leq 2 \text{ and } |z| \leq 1\}$ in the vertical string determined

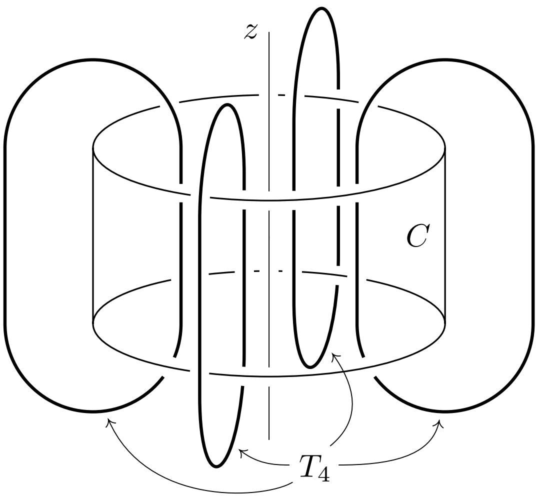

Figure 7.

by $\rho = 1$ (see Fig. 7). By using move J, we can obtain the special configurations for link diagrams described in the next proposition, where S^3 is thought as $R^3 \cup \{\infty\}$.

Proposition 4. *Let M be a closed orientable 3-manifold. Then for every n-link $L \subset M$ there exists $p \in \mathcal{S}(M)$ of any given topological type, such that $p(L) = T_n$ and $B_p \subset C$ is a closed braid around the z-axis. If L is a knot ($n = 1$), then T_1 can be replaced by the z-axis itself.*

Proof. Let $L \subset M$ any n-link. By Proposition 3 and isotopy of S^3, we obtain a p-diagram of L, with $p \in \mathcal{S}(M)$ of any given topological type, such that $p(L) = T_n$ and $B_p \subset C$. Then we modify p by using move J, in order to make the branching set B_p into a closed braid around the z-axis. Namely, we apply to B_p the Alexander-Birman algorithm (see Chap. 2 of [2]) with some minor modifications. After B_p has been oriented in any way, the algorithm consists of a finite sequence of tooth moves replacing negative edges with positive ones (see Fig. 8-a). By scaling along the z-axis we can assume that all these moves take place inside C. Whitout loss of generality, we can also assume that the triangle (tooth) involved in each move meets T_n at most in one point. Then, we modify any tooth which meets T_n as shown in Fig. 8-b, in order to realize it by a move J.

If $n = 1$, we can assume that $p(L)$ is the z-axis itself and apply to B_p the Alexander-Birman algorithm with all teeth modified as depicted in Fig. 8-c. $\square$

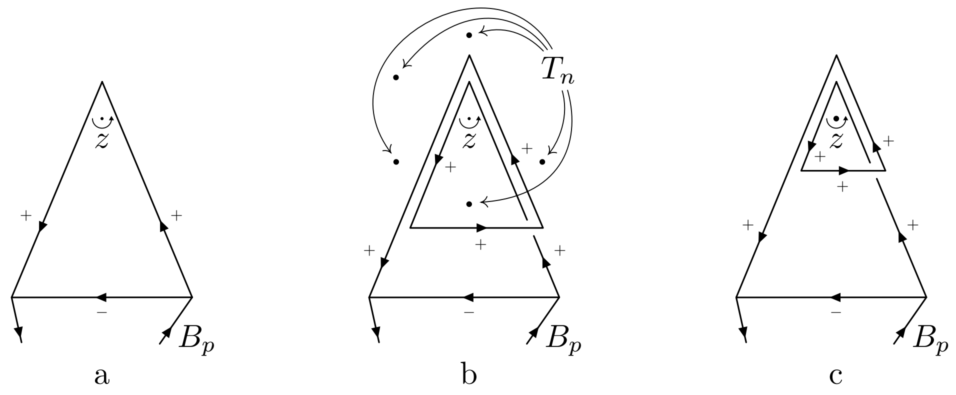

Figure 8.

Now, we adapt the technique introduced in [5], in order to obtain a universal configuration for link projections from the special one given by the previous proposition.

Proposition 5. *Let $L \subset M$ be any link in a closed orientable 3-manifold. Then there exists a branched covering $q : M \to S^3$ whose branching indices are 2 and 4, such that B_q is the Borromean link and $q(L)$ is parallel to a component of B_q (i.e. there exists an embedded annulus between $q(L)$ and such component, whose interior is disjoint from B_q).*

Proof. We start up by considering a 3-fold covering $p \in \mathcal{S}(M)$ with the properties stated in Proposition 4. Inside C we have (up to isotopy of S^3) the situation sketched in Fig. 9, where the horizontal braid representing B_p has to be thought trivially closed and at each crossing the horizontal string can run either under or over the vertical one contained in $p(L) = T_n$.

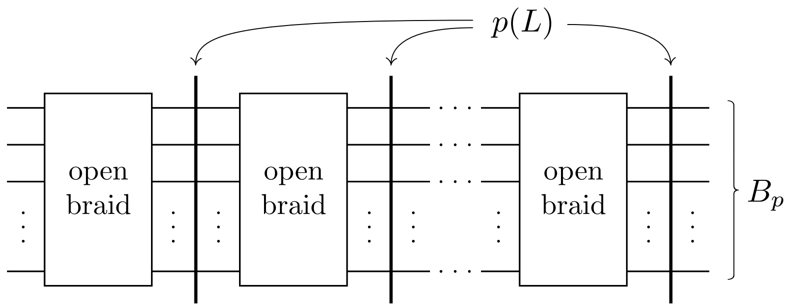

Figure 9.

Then we apply to $B_p \subset C$ all the modifications described in the proof of Theorem 1.1 of [5] up to Fig. 1.9 (note that the new strings added in Fig. 1.4 of [5] run under the link components). In this way, we get a new covering $p' \in \mathcal{S}(M)$ such that the part of the diagram depicted in Fig. 9 becomes as in Fig. 10.

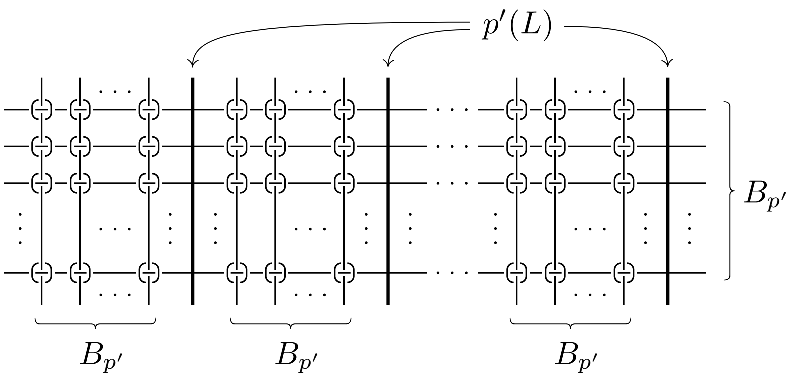

Figure 10.

Now we operate on each one of the crossings between $B_{p'}$ and $p'(L)$, in the following way: a) if $B_{p'}$ run under $p'(L)$ and their labels at the crossing are not disjoint, we insert a small trivial loop to $B_{p'}$ with monodromy $(i\,t)$, where the label t corresponds to an extra trivial sheet being added to the covering, in order to make the labels disjoint, and then invert the crossing by using move I (see Fig. 11); b) if $B_{p'}$ run under $p'(L)$ and their labels at the crossing are already disjoint, we directely invert the crossing by applying a move I and then insert a fake (with trivial monodromy) small loop to $B_{p'}$, in order to get the same configuration as above; c) if $B_{p'}$ run over $p'(L)$ we only insert a small fake branch loop around the crossing.

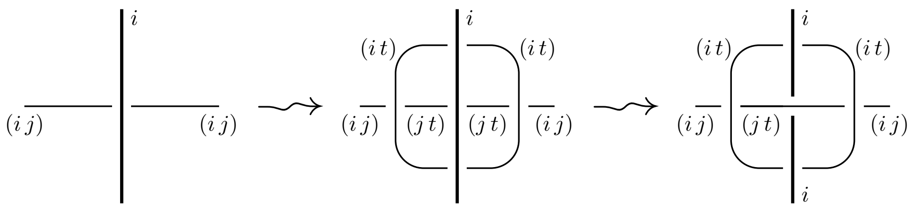

Figure 11.

At this point, we have a covering $p_1 \in \mathcal{S}(M)$ whose branching link has a "toroidal symmetry" as in Fig. 1.10 of [5], such that each component of $p_1(L)$ coincides with a fake meridian component of B_{p_1}. From now on, the proof of Theorem 1.1 of [5] (from Fig. 1.10 to the end) applies without any change, in order to get a (non-simple) covering $q' : M \to S^3$ with the following properties: a) $B_{q'}$ is a Borromean link, b) $q'(L)$ is a component of $B_{q'}$,

c) L is contained in the pseudo-singular link $S'_{q'}$, d) the branching indices of q' are 2 and 4.

We now change $q'(L)$ into a parallel curve, by a small locally unknotted isotopy. Property c) allows us to lift this isotopy to an isotopy of L in M, which can be realized by an ambient isotopy $H : M \times [0,1] \to M$. Then, the composition $q = q' h_1$ has the properties required in the statement. $\square$

Remark 6. Other universal configurations for link projections can be obtained starting from the one given by the previous proposition. In example, B_q can be any non-toroidal two-bridge link in S^3 and $q(L)$ any component of it (see [6]).

Applications to fibered links.

In this section we use generalized p-diagrams and moves, in order to give unified proofs (based on Lemma 7) of two different results on existence of fibered links in 3-manifolds. We recall that a link L in a closed orientable 3-manifold M is called a fibered link if it is the binding link of an open book decomposition of M.

First of all, we need to generalize our notion of link p-diagram, using multiple labels in order to represent any sublink L of the counterimage of a link $\overline{L} \subset S^3 - B_p$.

More precisely, given a p-diagram D of a 3-manifold M and a link $\overline{L} \subset S^3 - B_p$ such that the projection of $\overline{L}$ in the plane of D gives rise, together with D itself, to a diagram of $\overline{L} \cup B_p$, we represent $L \subset p^{-1}(\overline{L}) \subset M$ by labelling each overpass O_j of $\overline{L}$ with all the numbers of the sheets which meet $p^{-1}(O_j) \cap L$.

We also need to consider the generalized moves depicted in Fig. 12, where A and B are arbitrary multiple labels and A' corresponds to A under the transposition $(i\,j)$, and give conditions on the labels so that such moves induce isotopy or free homotopy on $L \subset M$.

It is easy to realize that: a) move I induces isotopy if $\{i,j\} \not\subset A$, while it induces only homotopy if $\{i,j\} \subset A$ (in this case L canghes into a new link $L' \subset M$ which is equivalent to L except for the clasp shown in Fig. 13); b) move II can be applied if and only if $\{i,j\} \cap A = \emptyset$ or $\{i,j\} \subset A$ and it induces isotopy in the first case and homology in the second one; c) move III induces isotopy if $A \cap B = \emptyset$, otherwise it induces homotopy.

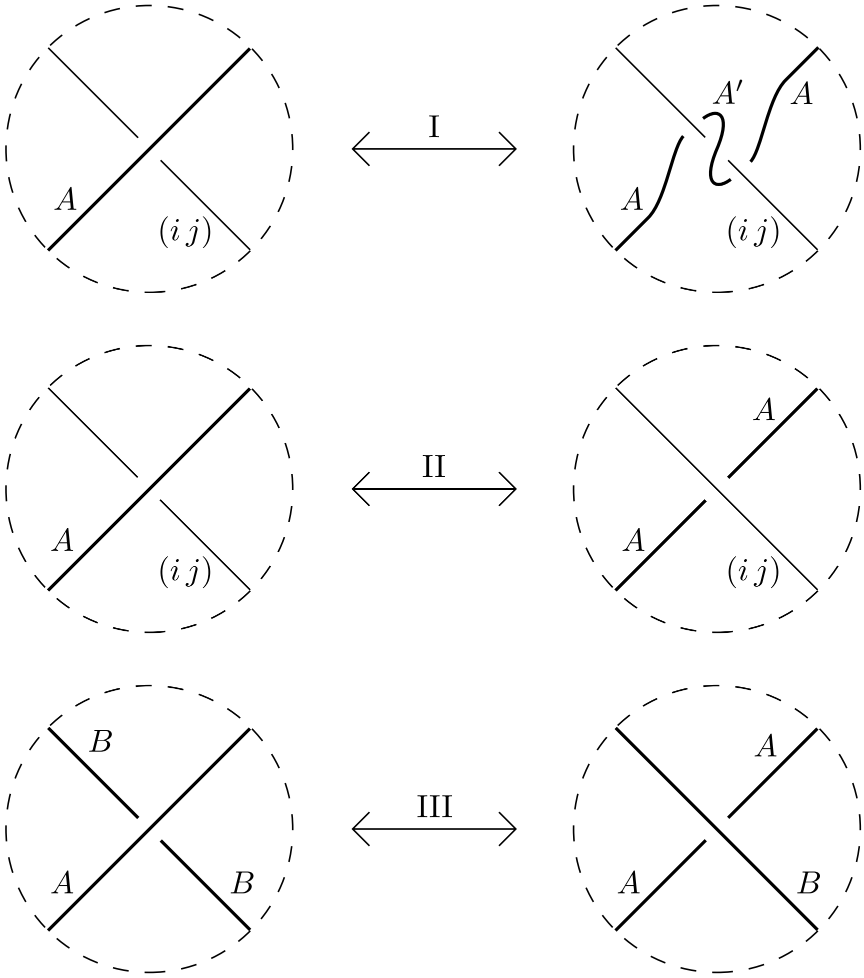

Figure 12.

Analogously, the generalized move J represented in Fig. 14 change a p-diagram of a link into a p'-diagram of a new link which is topologically equivalent to the original one if $\{i, j\} \not\subset A$ and homotopic to the original one if $\{i, j\} \subset A$.

By using move J, we prove the following lemma, where S^3 is thought as $R^3 \cup \{\infty\}$.

Lemma 7. *Let M be a closed orientable* 3*-manifold. Then, for any $p \in \mathcal{S}(M)$ such that $B_p \subset R^3 - z$-axis, there exists $p' \in \mathcal{S}(M)$ topologically equivalent to p such that $B_{p'}$ is a closed braid around the z-axis and the link $L' = p'^{-1}(z\text{-axis} \cup \{\infty\})$ is homotopically*

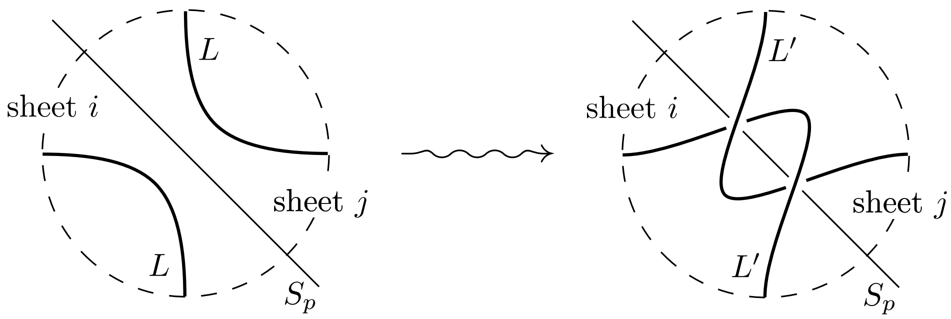

Figure 13.

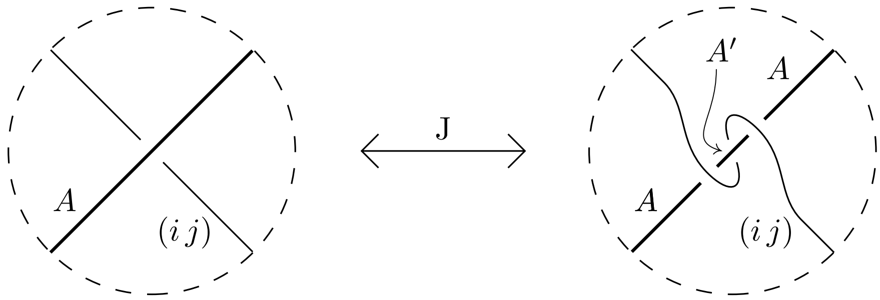

Figure 14.

equivalent to the link $L = p^{-1}(z\text{-axis} \cup \{\infty\})$*. Moreover, if* $L = L_1 \cup L_2$ *with* L_1 *and* L_2 *disjoint sublinks of* L*, then we can choose* $L' = L_1 \cup L'_2$ *with* L'_2 *isotopically equivalent to* L_2*.*

Proof. The first part of the lemma can be easily proved with the same technique used in the proof of Proposition 4. That is, given a covering p as in the statement, we make B_p into a braid around the z-axis, by means of the generalized move J described above. In this way we get a new covering p' with the required properties.

In order to prove the second part of the lemma, it is enough to show that, if $L = L_1 \cup L_2$, then B_p can be made into a braid around the z-axis, using only generalized moves J such that the labels i and j involved in the transposition do not correspond to the same sublink L_1 or L_2. In fact, in this way we get $L' = L'_1 \cup L'_2$, where L'_1 and L'_2 are respectively isotopically equivalent to L_1 and L_2. Hence, up to an isotopy relating L_1 to L'_1, we can assume $L'_1 = L_1$.

So, assume that, applying the algorithm described in the proof of Proposition 4, a move J is needed, such that the labels i and j involved in the transposition correspond to the same sublink L_1 or L_2 (see Fig. 15-a). Let k any label corresponding to the other sublink. Since the monodromy of p is transitive, starting from any positive edge of B_p, by using isotopy and, if necessary, a move J with transposition $(j\,k)$, we can obtain a positive edge of B_p labelled by $(j\,k)$ (as in Fig. 15-b), without introducing any new negative edge. Now, we can push part of this edge inside the cell where the original move J should take place (as shown in Fig. 15-c), in order to change the label A into A', where j and k are swapped. This makes the "bad" move J into a "good" one. □

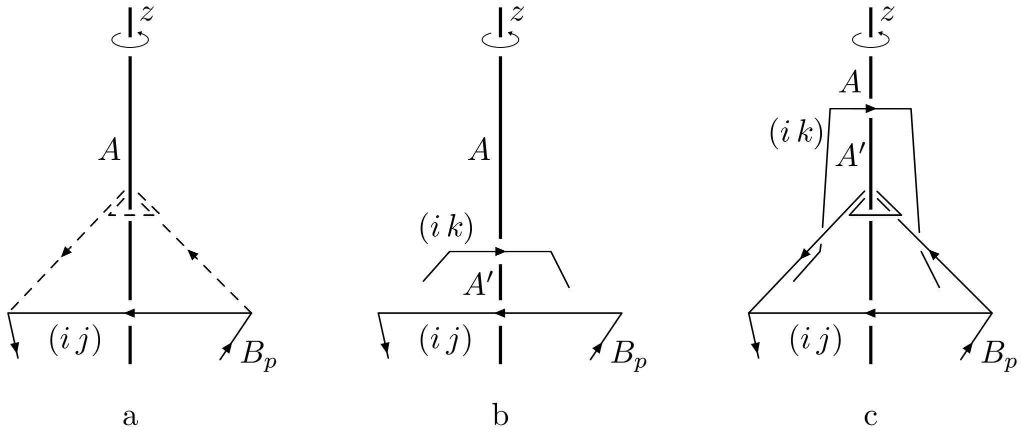

Figure 15.

We conclude with the following two propositions, which extend analogous results respectively obtained by Harer in [3] for knots and by Stallings in [13] for links in S^3.

Proposition 8. *Let $L \subset M$ be a link in a closed orientable 3-manifold. Then L is homotopically equivalent to a fibered link if and only if L is homologically trivial.*

Proof. The "only if" part is obvious. So we prove only the "if" part. Given a homologically trivial link $L \subset M$, there exists an orientable surface $F \subset M$ such that $\mathrm{Bd}\, F = L$. Let $q : F \to D$ be a simple d-fold covering (with $d \geq 3$) onto a trivial disk $D \subset S^3$. After q has been extended to regular neighborhoods, we apply Theorem 6.3 of [1] in order to extend it to a covering $p \in \mathcal{S}(M)$. Without loss of generality, we can assume that $\mathrm{Bd}\, D$ coincides with the z-axis $\cup \{\infty\} \subset R^3 \cup \{\infty\} \cong S^3$, hence $p^{-1}(z\text{-axis} \cup \{\infty\}) = L$. By using Lemma 7, we get a new covering $p' \in \mathcal{S}(M)$ such that $B_{p'}$ is a closed braid around the z-axis and $p'^{-1}(z\text{-axis} \cup \{\infty\}) = L'$ is a link homotopically equivalent to L. Now, the trivial open book decomposition of S^3 whose binding is the z-axis $\cup \{\infty\}$ lifts to an open book decomposition of M whose binding is L' (see Proposition 9.7 of [1]). So L' is a fibered link in M. □

Proposition 9. *Let $L \subset M$ be a link in a closed orientable 3-manifold. Then there exists a link $L' \subset M$ such that $L \cup L'$ is a fibered link. Moreover L' can be choosen in any isotopy class of links homologically equivalent to L.*

Proof. Let $L'' \subset M - L$ be any link homologically equivalent to L. As in the proof of Proposition 8, we can construct $p \in \mathcal{S}(M)$ such that $p^{-1}(z\text{-axis} \cup \{\infty\}) = L \cup L''$. By using Lemma 7, we get a new covering $p' \in \mathcal{S}(M)$ such that $B_{p'}$ is a closed braid around the z-axis and $p'^{-1}(z\text{-axis} \cup \{\infty\}) = L \cup L'$, with L' isotopically equivalent to L''. Then we conclude, as in the proof of Proposition 8, that $L \cup L'$ is a fibered link in M. □

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