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COVERING MOVES

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ABSTRACT

In this paper we give a positive answer to a long standing question posed by Montesinos, by introducing new covering moves, in order to relate any two colored diagrams representing the same 3-manifold as simple branched 3-covering of S^3.

Introduction

By a well-known theorem of Hilden and Montesinos ([6] and [7]), every closed orientable 3-manifold is a simple 3-covering of S^3 branched over a link.

Following R. Fox (cf. [5]), such a covering can be described by coloring each bridge of a planar diagram of the branch link with $R =$ red, $G =$ green or $B =$ blue, according to whether the monodromy of the corresponding meridian is the transposition $(1\,2)$, $(2\,3)$ or $(1\,3)$ of S_3.

Then we can use colored diagrams of links to represent 3-manifolds, and, of course, the natural question which immediately arises is to determine what different diagrams represent the same manifold. Montesinos long ago posed the problem, more or less in the following way:

> *Find a set of moves on colored link diagrams, such that any two diagrams represent the same manifold iff they are related (up to colored Reidemeister moves), by a finite sequence of such moves.*

The local move I, described in figure 1, was considered a possible answer for a long time (cf. [9]), up to when Montesinos found a counterexample (cf. [8]).

In this work we give a solution to the problem (equivalence theorem of section 1), by providing three new moves (see figure 7), which together with the previous one form a complete set of moves as required above.

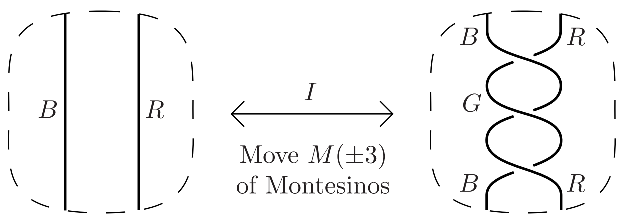


Figure 1.

That is obtained by normalizing the branch links as colored braids, corresponding to Heegaard spittings of the covering manifolds (cf. [1]), in such a way that we can apply results of [10] and [2], respectively in order to realize stable equivalence of splittings and to relate different braids representing the same splitting homeomorphism.

An application of our moves will be given in a forthcoming paper, in which we prove that any 4-manifold can be represented as a 4-fold branched covering of S^4.

0. Preliminaries

By an n-fold *branched covering* we mean a nondegenerate PL map $p : C \to X$ between compact PL manifolds, for which there exist subpolyhedra $S \subset C$ and $B \subset X$ (the *branch set*) of codimension 2, such that $p_{|C-S} : C - S \to X - B$ is an ordinary n-fold covering.

If B is a locally flat submanifold of X, then any n-fold ordinary covering of $X - B$ uniquely extends to an n-fold branched covering of X. Thus, n-fold branched coverings of X with branch set B are in 1-1 correspondence with the representations of $\pi_1(X-B)$ into S_n, the symmetric group of degree n.

For a branch link $L \subset S^3$, any such representation can be given by considering a planar diagram of L, and associating to each bridge of the diagram a permutation representing the corresponding Wirtinger generator of $\pi_1(S^3 - L)$. Moreover, in the case of a 3-fold branched covering, which is *simple* (that is all the Wirtinger generators are represented by transpositions), we can associate to each bridge of the diagram the color $R =$ red, $G =$ green or $B =$ blue, instead of the transposition $(1\,2)$, $(2\,3)$ or $(1\,3)$.

A link diagram whose bridges are colored in this way is called a *colored diagram*. Of course, the coloration of a colored diagram is not completely arbitrary, because of the Wirtinger relations. By directly substituting in any order the transpositions $(1\,2)$, $(2\,3)$, $(1\,3)$ for x, y, z in the typical form $x = yzy^{-1}$ of Wirtinger relators, we see that a coloration of a link diagram gives a colored diagram if and only if the colors of the three local arcs meeting at any overcrossing point are all the same or all distinct.

We define *colored Reidemeister moves*, by recoloring the part of the diagram modified by ordinary Reidemeister moves in the unique way compatible with the 'all the same or all distinct' rule. Then, a *colored isotopy* is given as a sequence of colored Reidemeister moves.

By a *normalized diagram* we mean a colored plat (= braid closed at the top and the bottom by simple arcs) diagram, such that all the top and bottom arcs are red except for the leftmost ones which are blue (cf. figure 2).

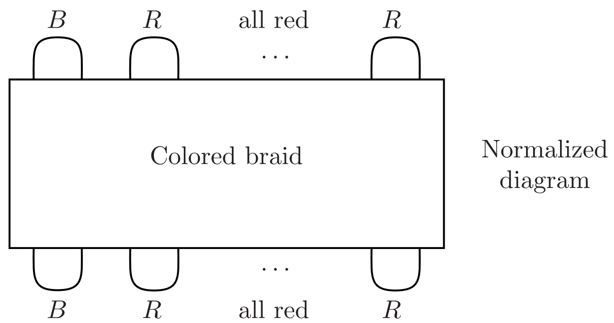


Figure 2.

Recall that an *m-braid* on S^2 is an element of the loop space $\Lambda(P_m S^2, *)$, where $P_m S^2$ denotes the set of all subsets of S^2 of order m, and $*$ is any such subset. Two braids are said to be *equivalent* if they are homotopy equivalent as based loops, that is if they represent the same element of the group $B(m) = \pi_1(P_m S^2, *)$, which is called the *braid group* of degree m of S^2.

A braid $b \in \Lambda(P_m S^2, *)$ can also be represented geometrically in $S^2 \times [0,1]$, by considering the union of all the sets $b(t) \times t$, with $t \in [0,1]$. Then, two braids are equivalent if and only if their geometric representations are isotopically equivalent in $S^2 \times [0,1]$ by means of an ambient isotopy fixing the boundary.

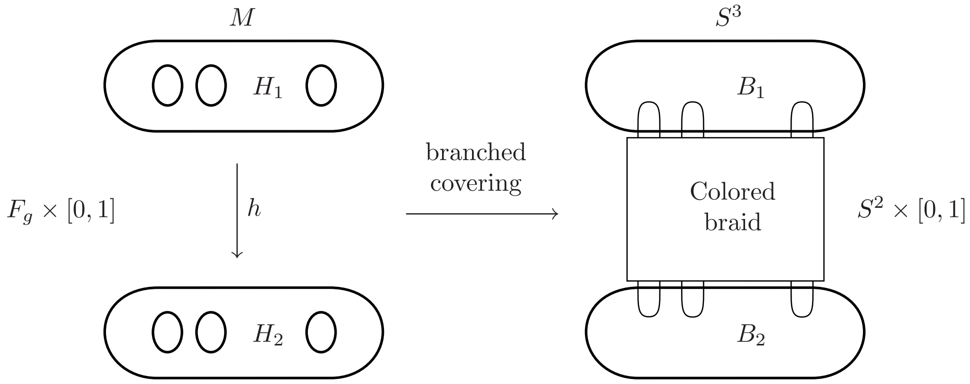


Figure 3.

Classical braids in R^3 between planes $z = z_0$ and $z = z_1$, such as the one in figure 2, can be seen as geometric representations of braids on S^2, considering that $R^2 \times [z_0, z_1] \cong R^2 \times [0,1] \subset S^2 \times [0,1]$, the inclusion being induced by the one-point compactification of R^2.

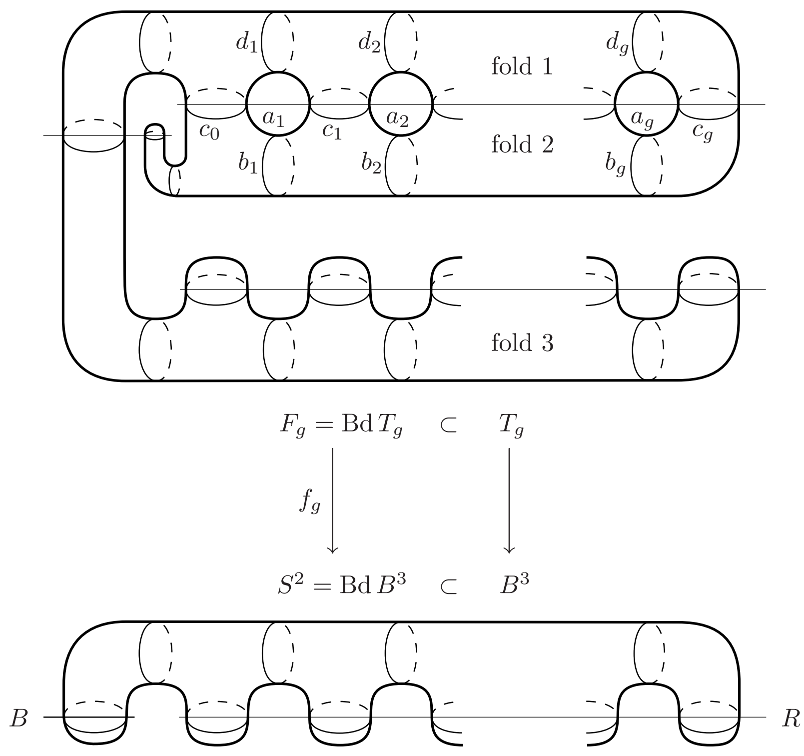


Figure 4.

Given any surface F, we denote by $M(F)$ the *mapping class group* of F, that is the group of all isotopy classes of homeomorphisms of F onto itself. $M(g)$ will be used instead of $M(F_g)$, where $F_g \subset R^3$ is the closed surface of genus g shown in figure 4.

Now, we say that an ambient isotopy $\{h_t\}_{0 \le t \le 1}$ of S^2 *realizes* a braid b if $h_t(*) = b(t)$, for every $0 \le t \le 1$. It is a classical result (cf. [3]), that any braid can be realized by an (obviously not unique) ambient isotopy of S^2, and that the correspondence $b \mapsto h_{1|S^2-*}$, induces a group epimorphism

$$\eta : B(g) \to M(S^2 - *).$$

Any normalized colored diagram induces a Heegaard splitting $M = H_1 \cup_h H_2$ of the covering manifold, as sketched in figure 3.

In order to describe the splitting, we observe that the handlebodies H_1 and H_2 cover (by restriction) the two 3-cells B_1 and B_2 in a standard way, which depends only on the genus of the splitting. Therefore we can identify H_1 and H_2 with a fixed handlebody $T_g \subset R^3$, for all the splittings of genus g induced by normalized diagrams. The covering after this identification is described in figure 4.

Let $f_g : F_g \to S^2$ be the restriction of such a covering to the boundary. Then the splitting homeomorphism $h : \mathrm{Bd}\, H_1 \to \mathrm{Bd}\, H_2$ is (up to isotopy) the lifting with respect to f_g of the homeomorphism h_1 of any isotopy which realizes the $(2g+4)$-braid of the diagram.

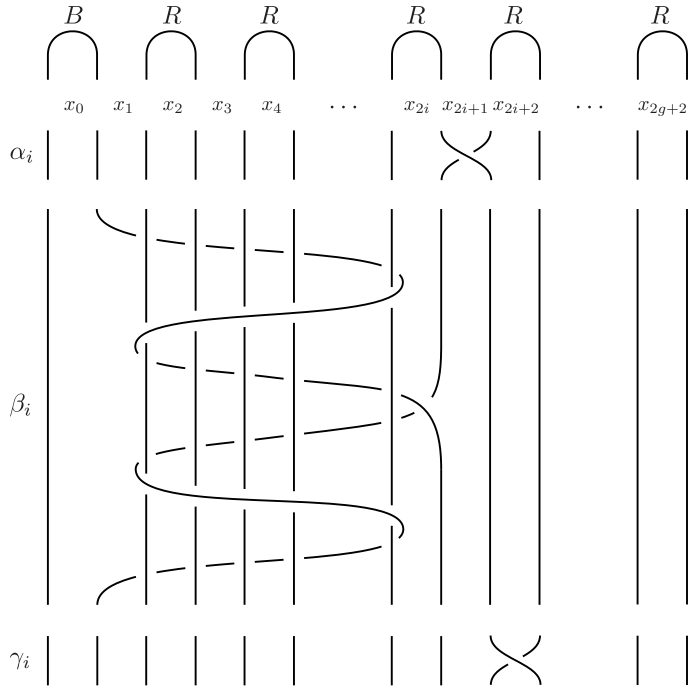


Figure 5.

We remark that this construction cannot be applied to any $(2g+4)$-braid b. In fact h_1 is liftable with respect to f_g if and only if b is colorable as the braid of a normalized diagram. In this case we say that b is *liftable*. By using colored isotopy, we see that liftability is invariant under equivalence of braids. Then we denote by $L(2g+4) \subset B(2g+4)$ the *liftable braid group* consisting of all liftable equivalence classes of braids. Moreover, since lifting can be obviously defined in terms of the epimorphism η, we have a lifting group homomorphism

$$\lambda : L(2g+4) \to M(g).$$

It turns out (cf. [1] or [4]) that any Heegaard splitting (hence any 3-manifold) can be represented in the above way, by means of a normalized diagram. This follows immediately from the fact that the homomorphism λ is onto. In figure 5 are shown braids representing (= lifting to) the left-handed Dehn twists α_i, β_i and γ_i about the loops a_i, b_i and c_i drawn in figure 4, which, by a well-known theorem of Lickorish (cf. [3]), generate the mapping class group $M(g)$.

Namely, if $x_0, \dots, x_{2g+2}$ are the standard generators for the braid group $B(2g+4)$ (cf. figure 5), and [x]y denote the braid $y^{-1}xy$ then:

- α_i is represented by x_{2i+1};
- β_i is represented by $[x_{2i}](x_{2i-1} \dots x_2 x_1^2 x_2 \dots x_{2i-2} x_{2i-1}^2 x_{2i-2} \dots x_1)$, in fact this braid corresponds to the half-twist about the arc $p_i \subset S^2$ shown in figure 6, whose lifting is, up to isotopy, β_i (cf. [2] for β_2);
- γ_i is represented by x_{2i+2}.

We warn the reader that, in drawing colored diagrams, we usually omit superfluous color labels; they can be deduced, by the 'all the same or all distinct' rule.

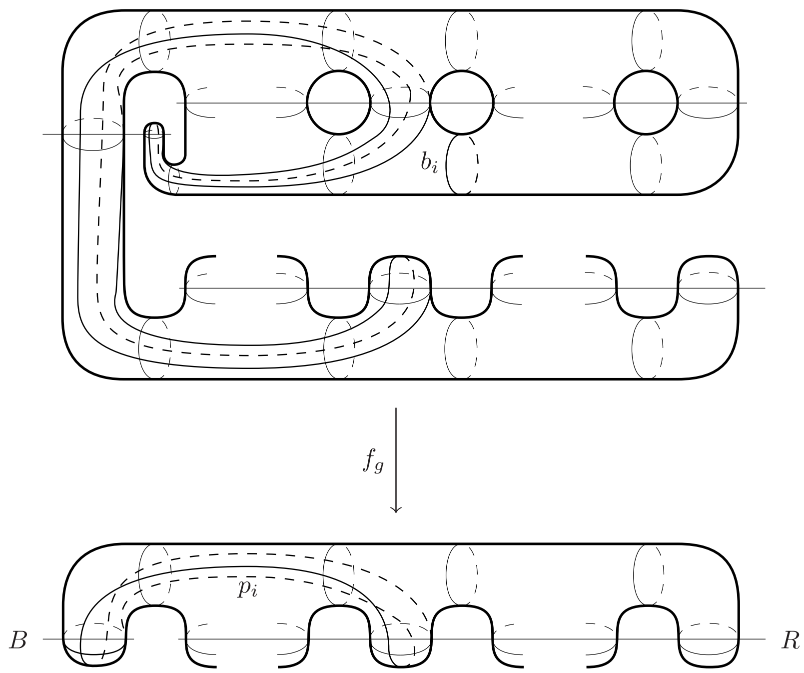


Figure 6.

1. The new moves and the equivalence theorem

Let us begin this section by describing the new moves we introduced, in order to relate colored diagrams representing the same manifold. We do that in figure 7, where all the diagrams consist of a colored braid (trivial on the left side) joining two arbitrary colored links L and L'.

We remark that such moves, in contrast with the Montesinos move, are not local. In fact they involve the global structure of the diagram (in figure 7 all the diagrams are completely drawn).

The fact that they do not change the covering manifold (up to homeomorphism) will be proved in section 3, by showing that all the braids on right side of figure 7 are equivalent to braids which represent the identity in $M(g)$.

Finally, we note that the new moves, together with the Montesinos move, might be not completely independent. In fact, we conjecture that this is the case, but at present we have no results in this direction.

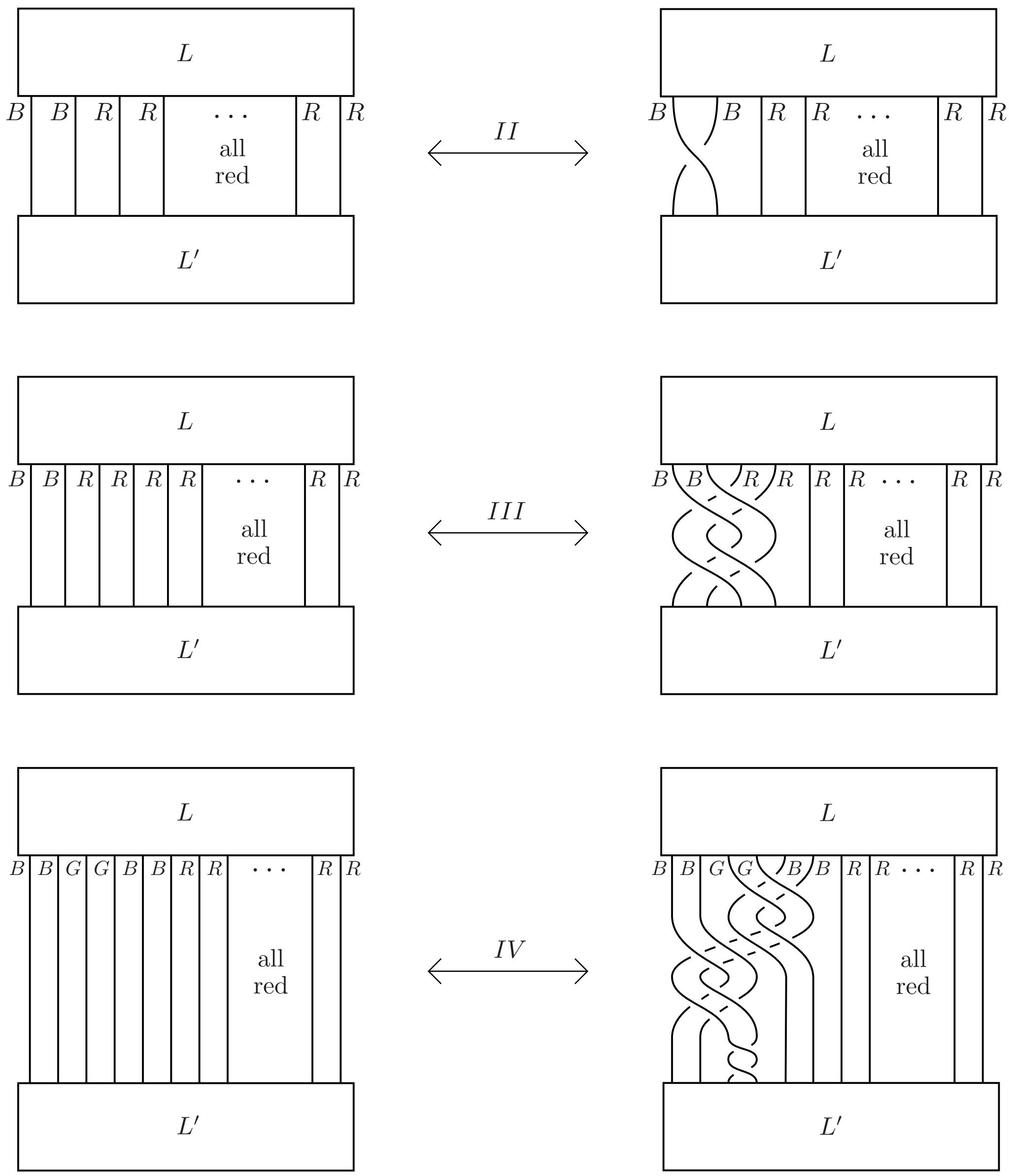


Figure 7.

Now, we can state the main result of our work:

The Equivalence Theorem. *Two colored link diagrams represent the same manifold iff they can be related (up to colored Reidemeister moves) by a finite sequence of moves of the types $I, \ldots, IV$ (described in figures 1 and 7).*

We prove the theorem in three steps: (1) normalization of any colored diagram by means of colored isotopy, that is colored Reidemeister moves; (2) realization of stable equivalence of Heegaard splittings represented by normalized diagrams, by using colored isotopy and move I; (3) connection of normalized diagrams representing the same splitting homeomorphism.

The first step is quite elementary and we conclude this section by proving it. The other two steps are more complicated and will be carried out respectively in sections 2 and 3.

In order to normalize a colored link diagram, we can assume that the diagram is a plat, since any link has a plat presentation (cf. [4]).

Then we need only to get the right colors for the top and bottom arcs of the plat. This can be done separately for the top and the bottom, by using the local isotopies shown in figure 8, if we assume (as we can easily do) that all the colors are present there. The first kind of isotopy allows us to reorder the colors, the second one can be used for eliminating the green arcs, and the third one makes it possible to eliminate all the blue arcs but the leftmost.

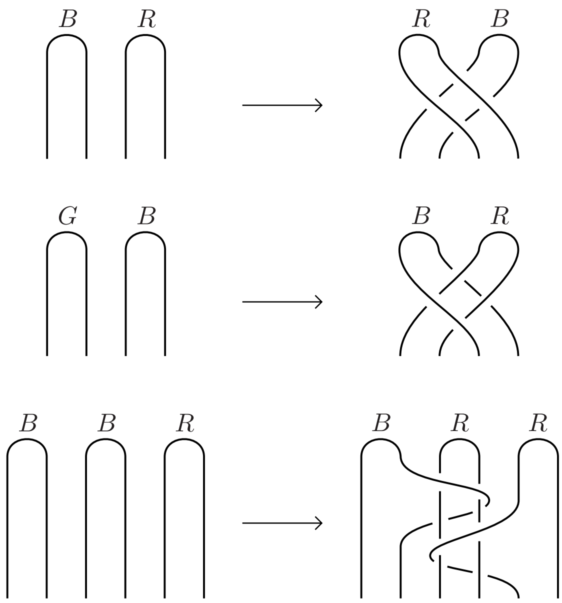


Figure 8.

2. Realizing stable equivalence of Heegaard splittings

In this section we prove that any two normalized diagrams representing the same manifold, can be modified by using colored isotopy and move I, in such a way that they induce the same splitting homeomorphism.

Let D and D' be two normalized diagrams representing the same 3-manifold M, then the corresponding Heegaard splittings $M = H_1 \cup_h H_2$ and $M = H_1' \cup_{h'} H_2'$ are stably equivalent (cf. [11]). That is, there exist stabilizations:

$$H_1'' \cup_{\widetilde{h}} H_2'' = (H_1 \cup_h H_2) \,\#\, (\#_n T \cup T)$$

and

$$H_1'' \cup_{\widetilde{h}'} H_2'' = (H_1 \cup_{h'} H_2) \,\#\, (\#_n T \cup T)$$

of them, where $T \cup T$ denotes the genus 1 splitting of S^3, and there exists a commutative diagram as follows:

$$\begin{tikzcd} H_1'' \supset \mathrm{Bd}\, H_1'' \arrow[r, "\widetilde{h}"] \arrow[d, "h_1"] & \mathrm{Bd}\, H_2'' \subset H_2'' \arrow[d, "h_2"] \\ H_1'' \supset \mathrm{Bd}\, H_1'' \arrow[r, "\widetilde{h}'"] & \mathrm{Bd}\, H_2'' \subset H_2'' \end{tikzcd}$$

where all the vertical arrow are homeomorphisms. (This means that the two homeomorphisms h_1 and h_2 between the boundaries of the handlebodies extend to homeomorphisms between the handlebodies.)

First of all, we observe that stabilization can be represented in terms of normalized diagrams, by iterating the modification described in figure 9 (which corresponds to make connected sum with a copy of $T \cup T$).

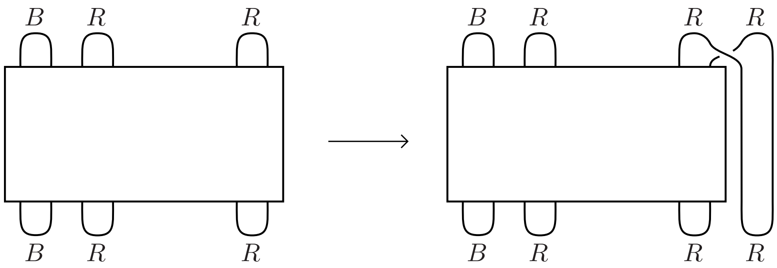


Figure 9.

Then, by means of colored isotopy, we can change D and D' into normalized diagrams $\widetilde{D}$ and $\widetilde{D}'$ representing the stabilized splittings $M = H_1'' \cup_{\widetilde{h}} H_2''$ and $M = H_1'' \cup_{\widetilde{h}'} H_2''$.

Now, if we add on top and on bottom of $\widetilde{D}$ two colored braids, respectively lifting to the homeomorphisms h_1^{-1} and h_2, we get a new normalized diagram inducing the same splitting homeomorphism $\widetilde{h}'$ induced by $\widetilde{D}'$.

The rest of this section is entirely devoted to show how to add these braids, by means of the prescribed moves.

Denote by $M^*(g)$ the subgroup of $M(g)$ consisting of all isotopy classes of the homeomorphisms of F_g onto itself, which extend to homeomorphisms of the handlebody T_g onto itself (cf. [10]).

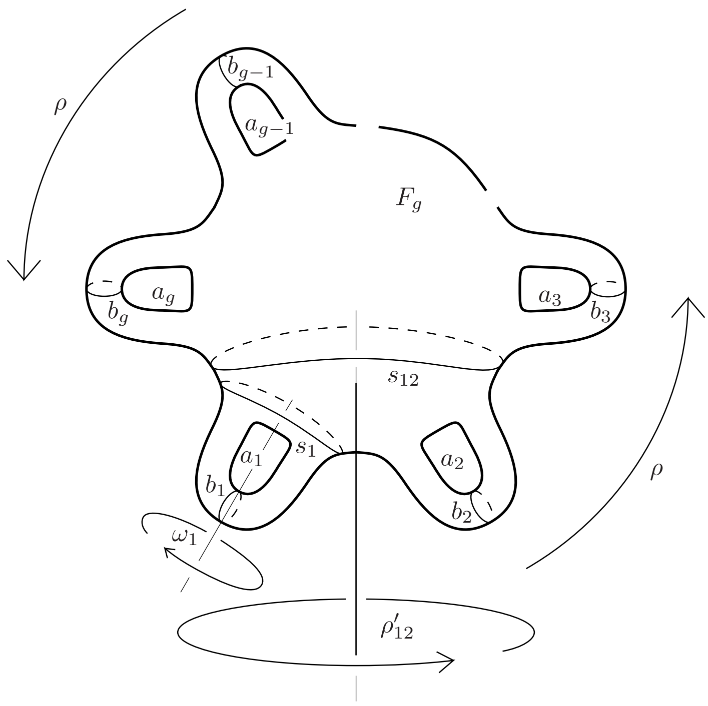


Figure 10.

We will prove that any element of $M^*(g)$ is represented by a colored braid which can be added on top (and hence on bottom) of a normalized diagram by using only colored isotopy and move I.

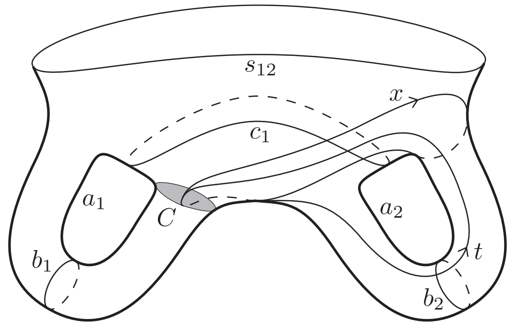


Figure 11.

In [10], Suzuki proved that $M^*(g)$ is generated by the isotopy classes of the following homeomorphisms (we drop the dots in Suzuki's notation):

$$\rho\,,\rho_{12}\,,\omega_1\,,\tau_1\,,\theta_{12} \text{ and } \xi_{12}\,,$$

where ρ is the cyclic permutation, by rotation of $2\pi/g$ radians, of the g knobs of F_g embedded in R^3 as in figure 10, and moreover, by looking at figures 10 and 11, and denoting by σ_i and $\sigma_{i,i+1}$ the Dehn twists about the loops $s_i = \rho^{i-1}(s_1)$ and $s_{i,i+1} = \rho^{i-1}(s_{12})$:

- τ_i is the right-handed twist of the i-th handle, that is β_i^{-1};
- ω_i is the half twist of the i-th knob, whose square is σ_i^{-1};
- $\rho_{i,i+1} = \rho'_{i,i+1}\omega_i\omega_{i+1}$ where $\rho'_{i,i+1}$ interchanges the i-th and $(i+1)$-th knobs, by means of a half twist whose square is $\sigma_{i,i+1}$;
- θ_{12} is (modulo β_1) the sliding of the attaching loop $\mathrm{Bd}\,C$ of the first handle (to S^2) about the oriented simple loop $t \subset F_g \cup C$;
- ξ_{12} is (modulo β_1) the sliding of the same attaching loop about the oriented simple loop $x \subset F_g \cup C$.

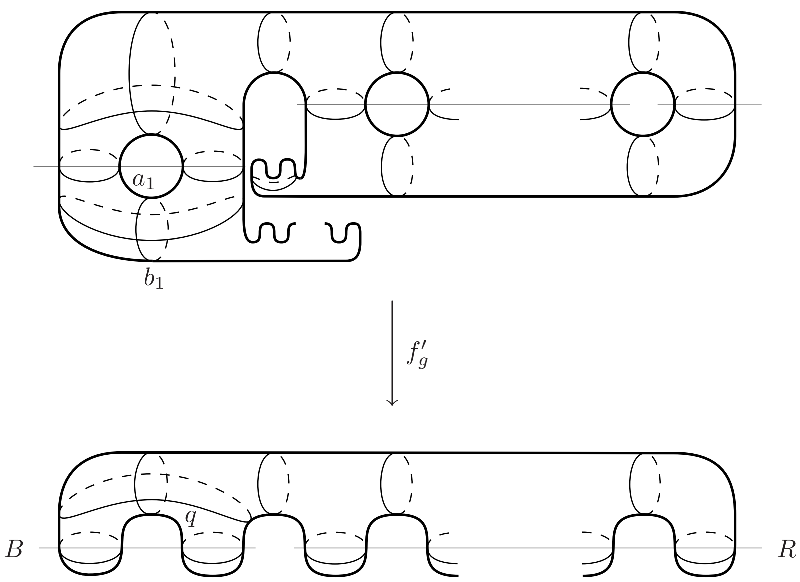


Figure 12.

Let us write all these homeomorphisms in terms of the standard generators of $M(g)$. First of all, since the loop x of figure 11 is (up to isotopy in $F_g \cup C$ modulo C) the core of the annulus $A \subset F_g \cup C$ bounded by $\omega_2^{-1}(c_1)$ on the left side (with respect to the orientation of x) and by $\omega_2^{-1}(b_2)$ on the right side, we easily get the sliding $\omega_2\gamma_1\beta_2^{-1}\omega_2^{-1}$ for ξ_{12}.

Analogously, by considering the loop t and the annulus $A' \subset F_g \cup C$ bounded by $(\alpha_2\beta_2)(b_2)$ on the left side and by $(\alpha_2\beta_2)(c_1)$ on the right side, we get the sliding $\beta_2^{-1}\alpha_2^{-1}\beta_2\gamma_1^{-1}\alpha_2\beta_2$ for θ_{12}.

Moreover, by looking at the branched covering $f'_g : F_g \to S^2$ of figure 12, we see that ω_1 is the lifting of a half twist of the disk containing the blue branch points and bounded by the simple loop q, and it can be represented by the (blue) braid $(x_0x_1x_2x_0x_1x_0)^{-1}$, which obviously also represent $(\alpha_1\beta_1)^{-3} = (\alpha_1^2\beta_1\alpha_1^2\beta_1)^{-1}$; so we have $\omega_1 = (\alpha_1^2\beta_1\alpha_1^2\beta_1)^{-1}$.

In the same way, by looking at the branched covering $f''_g : F_g \to S^2$ of figure 13, we see that ρ'_{12} is the lifting of a half twist of the disk containing the blue branch points and bounded by the simple loop r, and we have (by using standard identities for getting a more convenient form):

$$\begin{aligned}
\rho'_{12} &= \beta_1\alpha_1\gamma_1\alpha_2\beta_2\beta_1\alpha_1\gamma_1\alpha_2\beta_1\alpha_1\gamma_1\beta_1\alpha_1\beta_1 \\
&= \beta_1\alpha_1\gamma_1\alpha_2\beta_2\beta_1\alpha_1\beta_1\gamma_1\alpha_1\alpha_2\gamma_1\beta_1\alpha_1\beta_1 \\
&= \beta_1\alpha_1\gamma_1\alpha_2\beta_2\alpha_1\beta_1\alpha_1\gamma_1\alpha_1\alpha_2\gamma_1\beta_1\alpha_1\beta_1 \\
&= \beta_1\alpha_1\gamma_1\alpha_2\beta_2\alpha_1\beta_1\gamma_1\alpha_1\gamma_1\alpha_2\gamma_1\beta_1\alpha_1\beta_1 \\
&= \beta_1\alpha_1\gamma_1\alpha_2\beta_2\alpha_1\gamma_1\beta_1\alpha_1\beta_1\gamma_1\alpha_2\gamma_1\alpha_1\beta_1 \\
&= \beta_1\alpha_1\gamma_1\alpha_2\beta_2\alpha_1\gamma_1\alpha_1\beta_1\alpha_1\gamma_1\alpha_2\gamma_1\alpha_1\beta_1\,.
\end{aligned}$$

Finally, a straightforward computation of the actions on $\pi_1(F_g)$, shows that $\rho = \rho_{g-1,g}\rho_{g-2,g-1}\dots\rho_{12}\omega_1^2$; in fact, by referring to the notation of [10], we have that the induced homomorphisms coincide, up to conjugation by the commutator s_1.

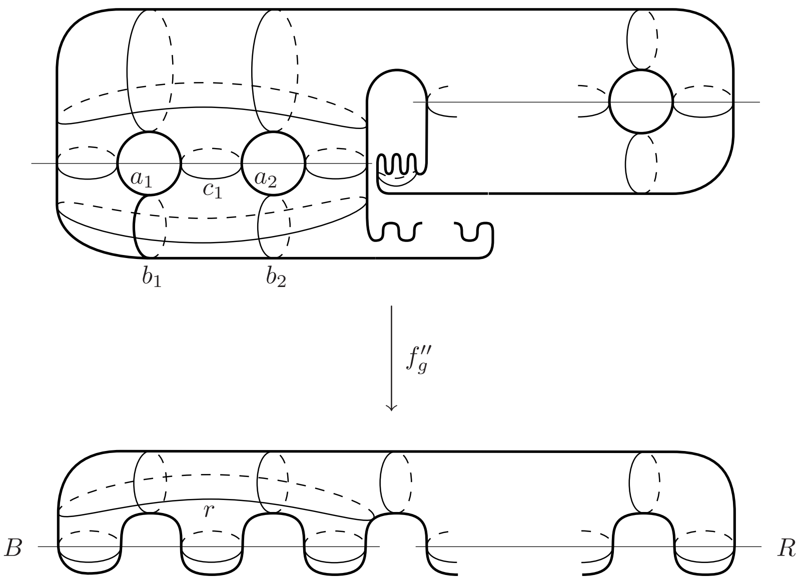
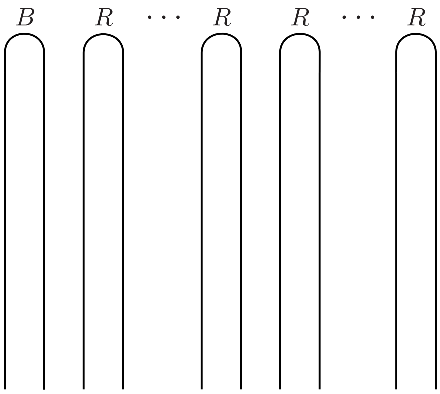
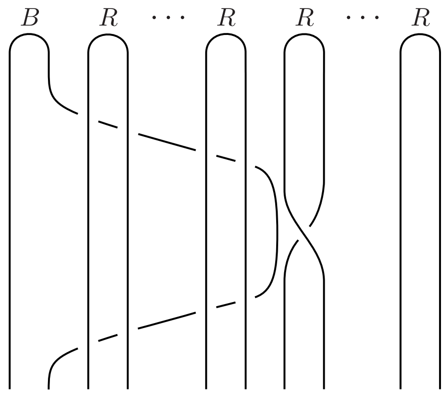


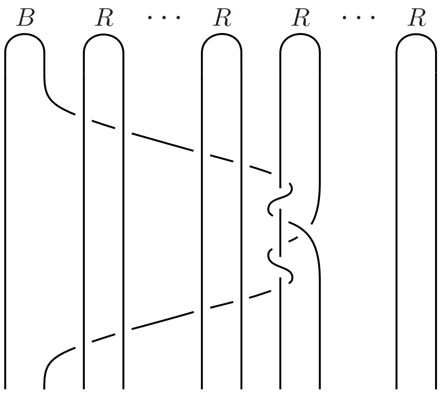
Figure 13.



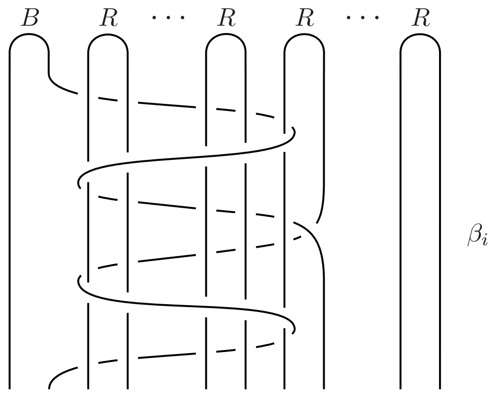
(a) Top of a normalized diagram



(b) . . . by colored isotopy

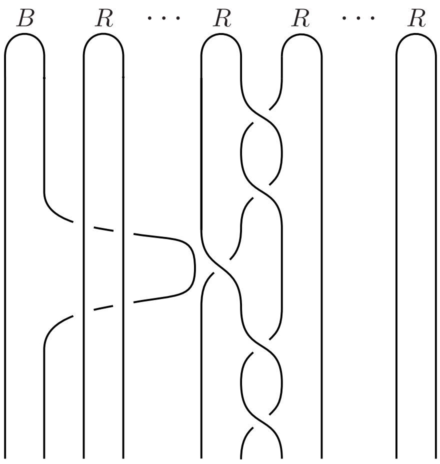


(c) . . . by two Montesinos moves

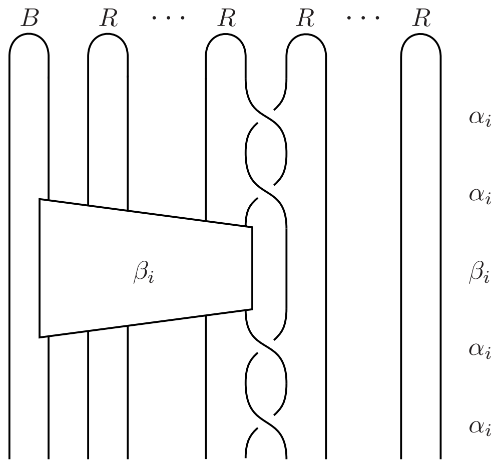


(d) . . . by colored isotopy

Figure 14.

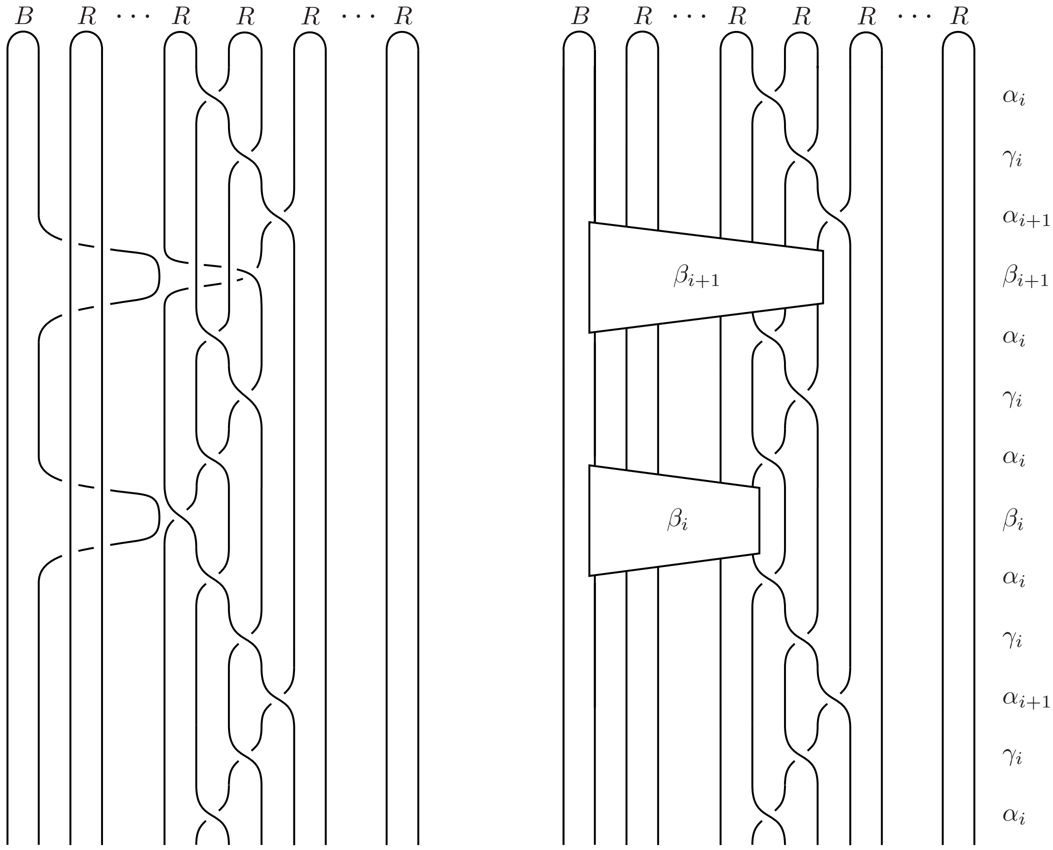


(a) By colored isotopy from the top



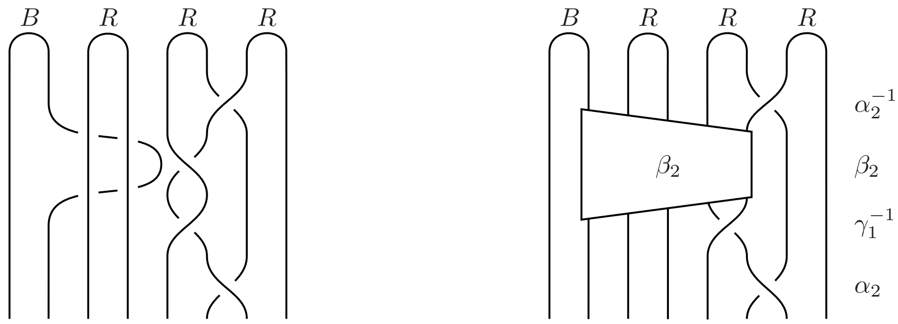
(b) . . . as in figure 14

Figure 15.



(a) By colored isotopy from the top (b) ... as in figure 14 (+ isotopy for β_{i+1})

Figure 16.



(a) By colored isotopy fromp the top (b) ... as in figure 14

Figure 17.

So, we have the following identities, the second and the fourth of which are obtained by conjugating by ρ^{i-1} the analogous ones given above for $i = 1$:

$$\tau_i = \beta_i^{-1}\ ,\ \omega_i = (\alpha_i^2\beta_i\alpha_i^2\beta_i)^{-1}\ ,\ \rho = \rho_{g-1,g}\rho_{g-2,g-1}\dots\rho_{12}\omega_1^2\ ,$$
$$\rho_{i,i+1} = \beta_i\alpha_i\gamma_i\alpha_{i+1}\beta_{i+1}\alpha_i\gamma_i\alpha_i\beta_i\alpha_i\gamma_i\alpha_{i+1}\gamma_i\alpha_i\beta_i \quad (\text{modulo } \omega_i\omega_{i+1})\ ,$$
$$\xi_{12} = \omega_2\gamma_1\beta_2^{-1}\omega_2^{-1} \quad \text{and} \quad \theta_{12} = \beta_2^{-1}\alpha_2^{-1}\beta_2\gamma_1^{-1}\alpha_2\beta_2 \quad (\text{modulo } \beta_1)\ .$$

Hence, it is clear that we need only to show how to add on top of a normalized diagram colored braids representing the homeomorphisms:

$$\beta_i \text{ and } \alpha_i^2\beta_i\alpha_i^2 \text{ for } i = 1,\ldots,g \ ,$$
$$\alpha_i\gamma_i\alpha_{i+1}\beta_{i+1}\alpha_i\gamma_i\alpha_i\beta_i\alpha_i\gamma_i\alpha_{i+1}\gamma_i\alpha_i \text{ for } i = 1,\ldots,g-1 \ ,$$
$$\gamma_1 \text{ and } \alpha_2^{-1}\beta_2\gamma_1^{-1}\alpha_2 \ .$$

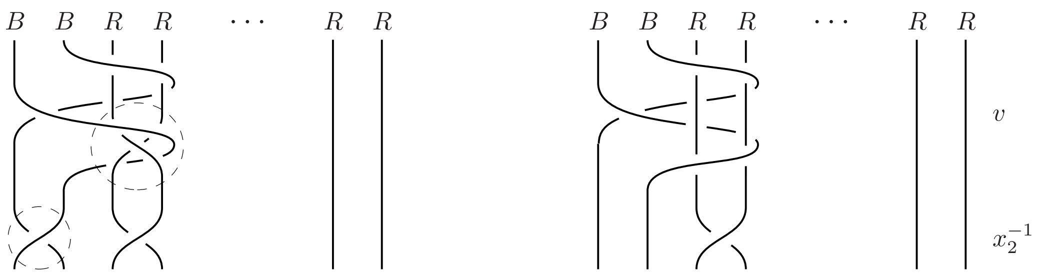
We do that (except for the trivial case of γ_1) in figures 14 to 17, which conclude this section.

3. Representing the identity splitting homeomorphism

In this section we prove that the new moves introduced in figure 7 do not change the covering manifold (up to homeomorphism), and then we use them to relate normalized diagrams inducing the same splitting homeomorphism.

Considering the homomorphism $\lambda : L(2g+4) \to M(g)$ defined in section 0, we have that two normalized diagrams induce the same splitting homeomorphism if and only if the corresponding braids differ by an element of $\ker \lambda$.

Then the two aims of this section can be achieved, by proving that: (1) all the braids on right side of figure 7 are (up to colored isotopy and Montesinos moves) in $\ker \lambda$; (2) elements of $\ker \lambda$ can be added on top of a normalized diagram by means of moves *I*–*IV*.



(a) By colored isotopy from the braid of move *III*

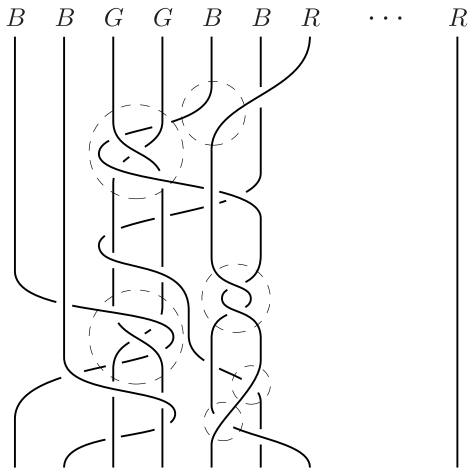
(b) ... by Montesinos moves inside the larger dashed circle in (a) and a move *II* inside the smaller one

Figure 18.

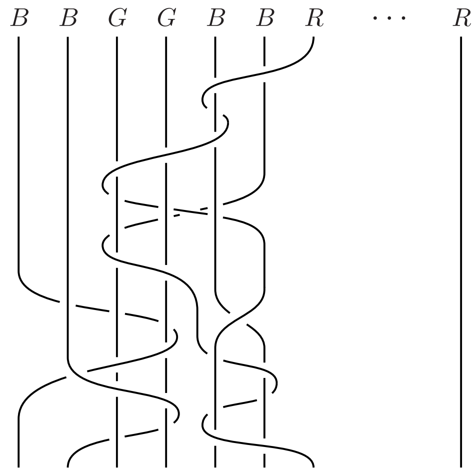
The first fact is well know for the braid x_0 corresponding to move *II* (cf. [1] or [4]). Moreover, it is easy to prove for the braid corresponding to move *III*, since this braid is equivalent, as shown in figure 18, to the braid vx_2^{-1}, where $v = [x_0](x_1x_2^2x_1)^{-1}$ represents the Dehn twist δ_1 about the loop d_1 drawn in figure 4 (cf. [1], where δ_1 is denoted by $t(b_1)$).

Finally, in figure 19 we show how to modify the braid corresponding to move *IV*, in order to get the braid $[u]y_2^{-1}w^{-1}$, where $u = [x_4](x_3x_2x_1^2x_2x_3^3x_2x_1)$, $y_2 = x_5x_4x_3x_2^2x_3x_4x_5$ and $w = (x_1x_2x_3x_4^2x_3x_2x_1)^{-2}$.

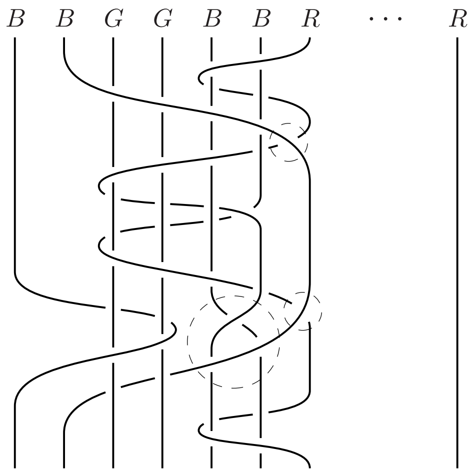
This last braid belongs to $\ker \lambda$, since w represents the Dehn twist δ_2 about the loop d_2 drawn in figure 4 (cf. [4], where δ_2 is denoted by β_2), u represents β_2 (cf. section 0) and the homeomorphism $\alpha_2\gamma_1\alpha_1\gamma_0^2\alpha_1\gamma_1\alpha_2$, given by y_2, makes b_2 into d_2 and viceversa.



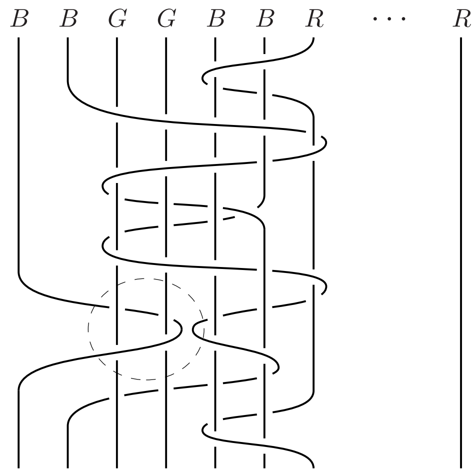
(a) By colored isotopy
from the braid of move IV



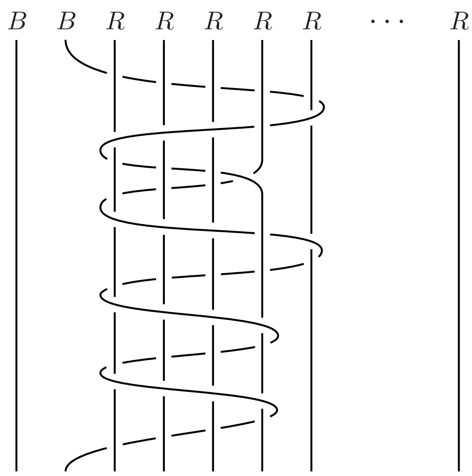
(b) . . . by Montesinos moves
inside the dashed circles in (a)



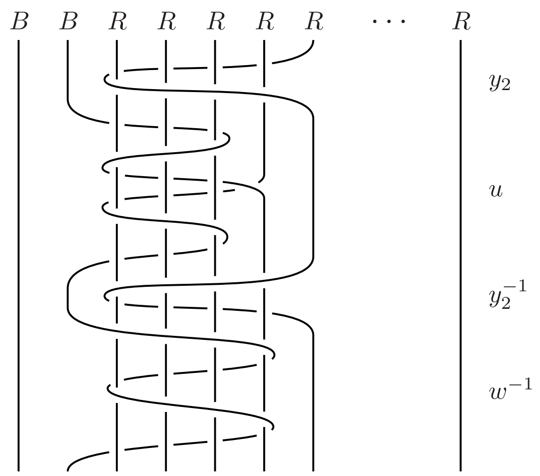
(c) . . . by colored isotopy



(d) . . . by Montesinos moves
inside the dashed circles in (c)



(e) . . . by conjugation and a move III
inside the dashed circle in (d)



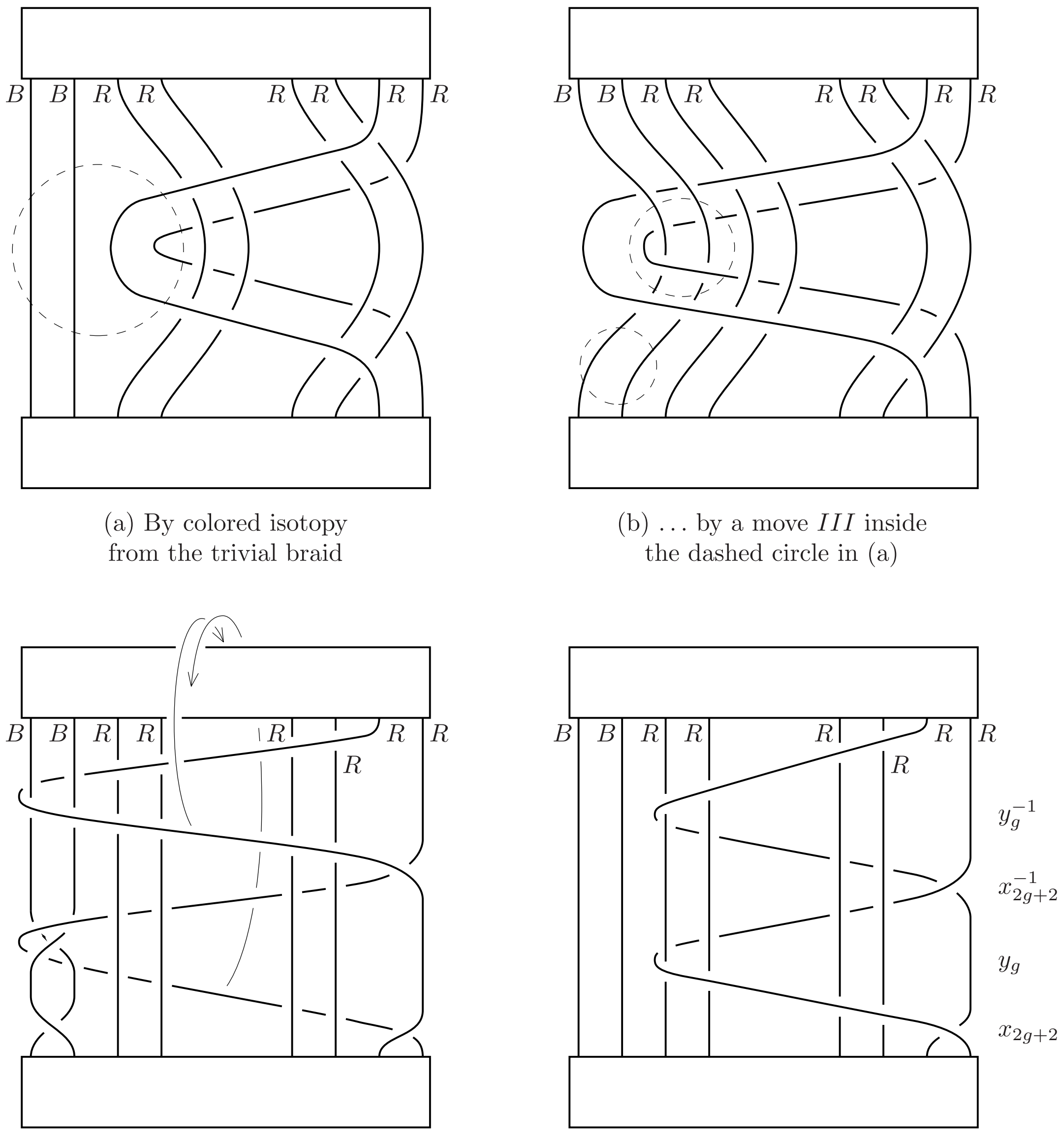
(f) . . . by colored isotopy

Figure 19.

In order to prove the second fact, let us consider the characterization of $\ker\lambda$ (given in [2], where the homomorphism λ is denoted by $\widetilde{\lambda}$) as the smallest normal subgroup of $L(2g+4)$ containing the following elements:

$$x_0 \quad , \quad x_1^3 \quad , \quad B' = (x_2x_3x_4)^4[u^{-1}]y_2^{-1}u^{-1} \quad , \quad y_g^{-1}x_{2g+2}^{-1}y_gx_{2g+2}$$

where $y_g = x_{2g+1}\dots x_3x_2^2x_3\dots x_{2g+1}$.



(a) By colored isotopy from the trivial braid

(b) … by a move III inside the dashed circle in (a)

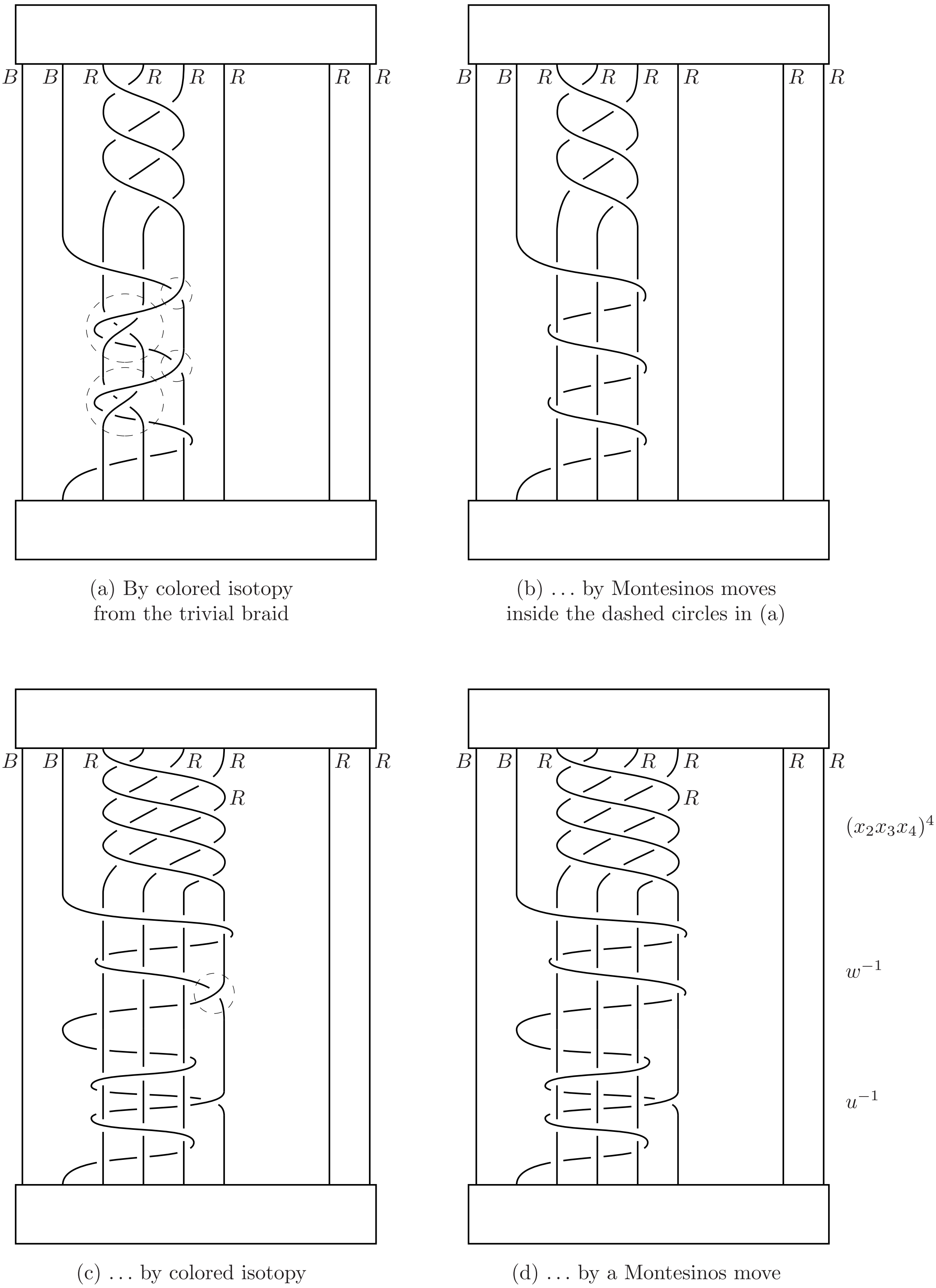
(c) … by Montesinos moves inside the larger circle in (b) and a move II inside the smaller one

(d) … by colored isotopy as shown in (c)

Figure 20.

Then, we are reduced to showing how to insert such four braids in a normalized diagram (that is something more than we really need, so this is a point where our work could be possibly improved).

For x_0 and x_1^3 we don't need to prove anything, since they can be obviously realized by move I and II. On the other hand, figure 20 shows how to get the braid $y_g^{-1}x_{2g+2}^{-1}y_gx_{2g+2}$.



(a) By colored isotopy from the trivial braid

(b) … by Montesinos moves inside the dashed circles in (a)

(c) … by colored isotopy

(d) … by a Montesinos move inside the dashed circle in (c)

Figure 21.

Figure 21 concludes the proof of the theorem, by showing how to realize the braid $(x_2x_3x_4)^4w^{-1}u^{-1}$, which is equivalent to B' up to moves, since what we have proved above (cf. figure 19).

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