BRANCHED COVERING REPRESENTATION OF NON-ORIENTABLE 4-MANIFOLDS

VALENTINA BAIS, RICCARDO PIERGALLINI, AND DANIELE ZUDDAS

Abstract. We show that every closed connected non-orientable PL 4-manifold X is a simple branched covering of $\mathbb{R}\mathrm{P}^4$. We also show that X is a simple branched covering of the twisted S^3-bundle $S^1 \mathbin{\widetilde{\times}} S^3$ if and only if the first Stiefel–Whitney class $w_1(X)$ admits an integral lift. In both cases, the degree of the covering can be any number $d \geqslant 4$, provided that d has the same parity of the Stiefel–Whitney number $w_1^4[X]$ in the case of $\mathbb{R}\mathrm{P}^4$. Moreover, the branch set can be assumed to be non-singular if $d \geqslant 5$ and to have just nodal singularities if $d = 4$.

Introduction

According to Piergallini–Iori [9, 10] any closed connected orientable PL 4-manifold X can be represented by a simple branched covering $p : X \to S^4$, whose degree $d(p)$ and branch set $B_p \subset S^4$ satisfy one of the following conditions:

(a) $d = 4$ and B_p is a locally flat self-transversally immersed PL surface,
(b) $d \geqslant 5$ and B_p is a locally flat embedded PL surface.

Furthermore, criteria for the existence of a simple branched covering $p : X \to N$ satisfying (a) or (b) with $N = \mathbb{C}\mathrm{P}^2, S^2 \times S^2, S^2 \mathbin{\widetilde{\times}} S^2$ were given by Piergallini–Zuddas [12] in terms of the Betti numbers $b_2^\pm(X)$.

Here, we address the representation of closed connected non-orientable PL 4-manifolds as branched covers of the basic non-orientable 4-manifolds $\mathbb{R}\mathrm{P}^4$ and $S^1 \mathbin{\widetilde{\times}} S^3$. Before stating our results, let us fix some notation and conventions.

Notation and conventions. We work in the PL category. However, our results hold in the DIFF category as well, being the two categories equivalent in dimension 4.

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All manifolds, submanifolds and maps between manifolds are assumed to be PL even if not explicitly specified. In particular the symbol $\cong$ stands for PL homeomorphism.

D^n will always denote an n-dimensional closed disk. $\mathbb{RP}^n$ is the n-dimensional real projective space and $S^1 \mathbin{\tilde{\times}} S^3$ is the twisted S^3-bundle over S^1.

For any locally flat submanifold $N \subset M$, we will indicate by $T(N, M)$ a closed regular neighbourhood of N inside M. In the following we will always have $\dim N = \dim M - 1$ or $\dim M \leqslant 4$, in which cases $T(N, M)$ is a D^k-bundle over N with $k = \dim M - \dim N$, namely it is equivalent to the disk-bundle associated to $\nu_M N$, the normal bundle of N in M. Hence, it makes sense to call $T(N, M)$ a closed tubular neighbourhood, to talk about the projection $T(N, M) \to N$, and to say that $T(N, M)$ is trivial or non-trivial.

Given an n-manifold M, $w_i(M) \in H^i(M; \mathbb{Z}_2)$ denotes its i^{th} Stiefel–Whitney class while $\mathrm{PD} : H^i(M; G) \to H_{n-i}(M; G)$ is the Poincaré duality isomorphism with coefficient group G with respect to a G-orientation (usually $G = \mathbb{Z}_2$ or $\mathbb{Z}$). An element of $H^i(M; \mathbb{Z}_2)$ is said to admit an integral lift if it lies in the image of the coefficient homomorphism $H^i(M; \mathbb{Z}) \to H^i(M; \mathbb{Z}_2)$.

Now we state our main theorems, which will be proved in Section 2.

Theorem A. *Given a closed connected non-orientable PL 4-manifold X and $d \geqslant 4$, there exists a d-fold branched covering $p : X \to \mathbb{RP}^4$ if and only if $d \equiv \langle w_1(M)^4, [X]\rangle \pmod 2$. Moreover, we can assume that p is simple and satisfies one of the above conditions* (a) *or* (b).

Theorem B. *Given a closed connected non-orientable PL 4-manifold X and $d \geqslant 4$, there exists a d-fold branched covering $p : X \to S^1 \mathbin{\tilde{\times}} S^3$ if and only if $w_1(X)$ admits an integral lift. Moreover, we can assume that p is simple and satisfies one of the above conditions* (a) *or* (b).

According to Berstein–Edmonds [2, Appendix], R. H. Fox showed in unpublished work that every closed connected non-orientable $2n$-dimensional PL manifold admits a branched covering over $\mathbb{RP}^{2n}$, for every $n \in \mathbb{N}$. Their proof is based on an equivariant version of the classical theorem of Alexander [1], and hence no control is given on the degree and on the regularity of the branch set.

We expect that Theorems A and B can be generalized to branched coverings of the connected sums $\#_m \mathbb{RP}^4 \#_n S^1 \mathbin{\tilde{\times}} S^3$ with $m + n \geqslant 1$, in the spirit of [12, Theorem 1.2]. We guess that the existence of such branched coverings can also be expressed in terms of certain algebraic constraints.

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1. Preliminaries and technical results

In this section we develop the technical results needed for the proofs of the Theorems A and B. Let us start by recalling the definition of branched covering in the PL category.

Definition 1.1. Given two compact n-manifolds M and N, a PL map

$$p : M \xrightarrow{d:1} N$$

is called a d-fold *branched covering* if it is non-degenerate and restricts to a d-fold ordinary covering over the complement of a codimension two closed subpolyhedron of N. The *branch set* of p is the smallest subpolyhedron $B_p \subset N$ with such property, while the *degree* $d = d(p)$ of p is the cardinality of the preimage of any point in $N \setminus B_p$.

Moreover, p is called a *simple* branched covering, if the preimage of a generic point of B_p has cardinality $d-1$, containing only one singular point of p, at which p has local degree 2.

A d-fold branched covering as above is fully determined up to PL homeomorphisms by the pair (N, B_p) and the *monodromy* representation

$$\omega_p : \pi_1(N \setminus B_p) \to S_d$$

associated to the ordinary covering $p| : M \setminus p^{-1}(B_p) \xrightarrow{d:1} N \setminus B_p$, where S_d denotes the permutation group of $\{1, \ldots, d\}$.

If N is simply connected, the fundamental group $\pi_1(N \setminus B_p)$ is generated by a suitable set of meridians of B_p and the monodromy representation ω_p is determined by the assignment of a permutation to each of these meridians so that the relations in $\pi_1(N \setminus B_p)$ are satisfied. For $\dim N \leqslant 4$ this is usually encoded by choosing a Wirtinger set of meridians with respect to some projection and by labeling each part of the projection diagram of B_p with the monodromy of the corresponding meridian.

However, if N is not simply connected (e.g. if $N = \mathbb{R}\mathrm{P}^n$) the situation is slightly more complicated, since meridians of B_p do not generate $\pi_1(N \setminus B_p)$ and one has to take into account also the monodromy of the generators of $\pi_1(N)$.

Moreover, if B_p is singular then the local monodromy at any singular point of B_p is subject to further constraints, in order to guarantee that the covering space M is a PL manifold.

In all cases, p is a simple branched covering if and only if the monodromy of each meridian of B_p is a transposition.

Finally, we observe that for every branched covering $p : M \to N$ the equality $p^*(w_1(N)) = w_1(M)$ holds thanks to the fact that B_p has codimension two in N.

Decomposing a non-orientable manifold

Here, we recall a standard way to decompose a non-orientable manifold into suitable pieces. Such a decomposition turns out to be useful in building branched coverings of non-orientable manifolds by using tools from the orientable setting.

Lemma 1.2. *For any closed connected non-orientable smooth n-manifold M, there is a closed connected orientable smooth $(n-1)$-submanifold $N \subset M$ such that $[N] = \mathrm{PD}(w_1(M))$ and $M' = M \setminus \mathrm{Int}\, T(N, M)$ is also connected and orientable. Moreover, N can be chosen in such a way that $T(N, M)$ is trivial if and only if $w_1(M) \in H^1(M; \mathbb{Z}_2)$ admits an integral lift.*

Proof. Let $N \subset M$ be a closed $(n-1)$-manifold representing the $\mathbb{Z}_2$-Poincaré dual of the first Stiefel–Whitney class of M (cf. [13, Théorème II.26]), i.e.

$$[N] = \mathrm{PD}(w_1(M)) \in H_{n-1}(M; \mathbb{Z}_2).$$

Up to a standard tubing argument, we can assume that N is connected. It easily follows that both N and $M' = M \setminus \mathrm{Int}\, T(N, M)$ are orientable and M' is connected.

We are left to show that N can be taken with trivial tubular neighbourhood if and only if $w_1(M)$ has an integral lift.

Indeed, whenever there is a class $\widetilde{w}_1(M) \in H^1(M; \mathbb{Z})$ such that $\widetilde{w}_1(M) \equiv w_1(M) \pmod 2$, one can find a smooth map $f : M \to S^1$ such that $f^*(\sigma) = \widetilde{w}_1(M)$, where $\sigma \in H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ is a preferred generator. The preimage of any regular value $v \in S^1$ will give an $(n-1)$-manifold $N \subset M$ with trivial tubular neighbourhood, which we can assume to be connected up to tubing. Moreover, N is orientable since it represents $\mathrm{PD}(w_1(M)) \in H_{n-1}(M; \mathbb{Z}_2)$.

For the converse, suppose that N has a trivial tubular neighbourhood. An integral lift of $w_1(M)$ is represented by the map $f : M \to S^1$ obtained from the projection $\pi : T(N, M) \cong N \times D^1 \to D^1$ by composing with the quotient map $D^1 \to D^1/\{\pm 1\} \cong S^1$ and extending such composition with the constant value $[\pm 1]$ on $M \setminus T(N, M)$. □

Remark 1.3. In Lemma 1.2, $\partial M' = \partial T(N, M)$ has two or one components depending on whether $T(N, M)$ is trivial or not, respectively. In both cases, there exists an orientation-preserving involution φ on $\partial M'$ such that $M \cong M'/\varphi$. The map φ is the non-trivial deck transformation of the 2-fold covering $\partial M' = \partial T(N, M) \to N$ given by the restriction of the projection map $T(N, M) \to N$. In particular, if $T(N, M)$ is trivial then $\partial M'$ is oriented as the boundary of M' with any orientation and φ interchanges its two components.

In the proof of Theorem A (resp. B), we will decompose the 4-manifold X into pieces as in Lemma 1.2. After that, the branched covering $p : X \to \mathbb{R}\mathrm{P}^4$ (resp. $S^1 \tilde{\times} S^3$) will be constructed starting with a branched covering $Y \to \mathbb{R}\mathrm{P}^3$ (resp. S^3) and extending it first to $T(Y, X)$ and then to the whole ambient 4-manifold X.

Stabilizing branched coverings over $\mathbb{R}\mathrm{P}^n$

Recall that a d-fold branched covering $p : M \to S^n$ (resp. D^n), can be stabilized to a $(d{+}1)$-fold branched covering $p' : M \to S^n$ (resp. D^n) by adding an extra trivial sheet. In terms of labeled branch set, this means that $B_{p'}$ is obtained from B_p by adding a separated unknotted $(n-2)$-sphere (resp. proper trivial $(n-2)$-disk) with monodromy $(i\ \ d{+}1)$ for some $i \in \{1, \dots, d\}$. The situation is rather different in the case of branched coverings over $\mathbb{R}\mathrm{P}^n$, as explained here below.

In the following, for $n \geqslant 2$ we will identify $T(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$ with $\mathbb{R}\mathrm{P}^n \setminus \mathrm{Int}\, D^n$, where $D^n \subset \mathbb{R}\mathrm{P}^n$ is an n-disk. Moreover, we note that $T(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$ is homeomorphic to a closed tubular neighbourhood of the zero section in the non-trivial line bundle over $\mathbb{R}\mathrm{P}^{n-1}$.

Lemma 1.4. *Let M be a compact connected n-manifold and*

$$p : M \xrightarrow{d:1} T(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$$

be a d-fold branched covering. Then there exists a $(d{+}2)$-fold branched covering

$$p' : M \setminus \mathrm{Int}\, D^n \xrightarrow{d+2:1} T(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n),$$

where $D^n \subset \mathrm{Int}\, M$ is a n-disk, the branch set $B_{p'}$ is the union of B_p and a separated proper trivial $(n-2)$-disk with monodromy $(i\ \ d{+}1)$ for some $i \in \{1, \dots, d\}$, and the monodromy of $\mathbb{R}\mathrm{P}^1$ is given by

$$\omega_{p'}([\mathbb{R}\mathrm{P}^1]) = \omega_p([\mathbb{R}\mathrm{P}^1]) \cdot (d{+}1\ \ d{+}2).$$

Proof. Let $u : S^n \to \mathbb{R}\mathrm{P}^n$ be the universal covering map and let

$$p_1 = p \sqcup u| : M \sqcup T(S^{n-1}, S^n) \longrightarrow T(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$$

be the $(d{+}2)$-fold branched covering obtained by adding to p two sheets with total space a u-equivariant tubular neighbourhood $T(S^{n-1}, S^n) \cong$

$S^{n-1} \times D^1$. Denote by

$$p_2 : \sqcup_{d+1} D^n \longrightarrow D^n$$

the $(d+2)$-fold branched covering of D^n whose branch set is a proper trivial $(n-2)$-disk with monodromy $(i\ \ d{+}1)$ for a fixed $i \in \{1, \dots, d\}$. Then, the branched covering p' is defined by taking the equivariant boundary connected sum of p_1 and p_2. Namely, such operation corresponds to performing a boundary connected sum of $T(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$ and D^n by attaching a suitable band between them, which at the level of the covering space amounts to attaching a lift of such band for every sheet. The conclusion follows by noticing that $M \natural (S^{n-1} \times D^1) \cong M \setminus \mathrm{Int}\, D^n$. □

As a consequence, we get the next proposition.

Proposition 1.5. *Let M be a closed connected n-manifold and*

$$p : M \xrightarrow{d:1} \mathbb{R}\mathrm{P}^n$$

a d-fold branched covering. Then, p can be stabilized to a $(d+2)$-fold branched covering

$$p' : M \xrightarrow{d+2:1} \mathbb{R}\mathrm{P}^n,$$

where the branch set $B_{p'}$ is the union of B_p and a separated trivial $(n-2)$-sphere with monodromy $(i\ \ d{+}1)$ for some $i \in \{1, \dots, d\}$, and the monodromy of $\mathbb{R}\mathrm{P}^1$ is given by

$$\omega_{p'}([\mathbb{R}\mathrm{P}^1]) = \omega_p([\mathbb{R}\mathrm{P}^1]) \cdot (d{+}1\ \ d{+}2).$$

Proof. Let $D^n \subset \mathbb{R}\mathrm{P}^n$ be an n-disk disjoint from B_p, so that $p^{-1}(D^n) = \sqcup_d D^n$, the disjoint union of d n-disks, and p is trivial over D^n. Then, we consider the restriction

$$p| : M \setminus \sqcup_d D^n \xrightarrow{d:1} T(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$$

and stabilize it according to Lemma 1.4 to a $(d{+}2)$-fold simple branched covering

$$p_1 : M \setminus \sqcup_{d+1} D^n \xrightarrow{d+2:1} T(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n).$$

Furthermore, we consider the $(d+2)$-fold branched covering

$$p_2 : \sqcup_{d+1} D^n \xrightarrow{d+2:1} D^n$$

whose branch set is a proper trivial $(n-2)$-disk with monodromy $(i\ \ d{+}1)$ for some $i \in \{1, \dots, d\}$. Finally, the statement follows by gluing p_1 and p_2 along the boundary, that is by putting $p' = p_1 \cup_\partial p_2$. □

Remark 1.6. In Proposition 1.5, we have

$$p^* = (p')^* : H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \to H^*(M; \mathbb{Z}_2).$$

This follows from the construction of the stabilization in Lemma 1.4 and Proposition 1.5, using also the fact that $d(p) \equiv d(p') \pmod 2$.

Even though we will not use this in the present paper, we notice that the proof of Proposition 1.5 works in a more general context. Given d-fold branched covering

$$p : M \xrightarrow{d:1} N$$

such that N admits a k-fold branched covering

$$q : S^n \xrightarrow{k:1} N,$$

we can stabilize p to a $(d+k)$-fold branched covering

$$p' : M \xrightarrow{d+k:1} N.$$

Namely, the branch set of p' is $B_{p'} = B_p \cup B_q \cup S^{n-2}$ where $S^{n-2} \subset N$ is a separated trivial $(n-2)$-sphere with monodromy $(d\ d{+}1)$ and the monodromy representation of p' satisfies

$$\omega_{p'}([\gamma]) = \omega_p([\gamma]) \cdot \omega_q([\gamma])$$

for any $[\gamma] \in \pi_1(N \setminus (B_p \cup B_q))$, where ω_q takes values in the permutation group of $\{d+1, \dots, d+k\}$.

Constructing branched coverings over $T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4)$

Let us start with some basic construction of simple branched coverings over the real projective plane.

Lemma 1.7. *Let Σ be a connected closed surface. For every $d \geqslant 2$ such that $d \equiv \chi(\Sigma) \pmod 2$ there is a d-fold simple branched covering*

$$p : \Sigma \xrightarrow{d:1} \mathbb{R}\mathrm{P}^2.$$

In particular, $p^*(w_1(\mathbb{R}\mathrm{P}^2)) = w_1(\Sigma)$.

Proof. If Σ is a 2-sphere, we build a d-fold simple branched covering $p : S^2 \xrightarrow{d:1} \mathbb{R}\mathrm{P}^2$ by stabilizing $d/2-1$ times the universal covering map $u : S^2 \xrightarrow{2:1} \mathbb{R}\mathrm{P}^2$ as in Lemma 1.4.

Suppose now that Σ is not a 2-sphere. If Σ is orientable, let $\gamma \subset \Sigma$ be any non-separating simple loop; otherwise, if Σ is non-orientable, we apply Lemma 1.2 to $M = \Sigma$ and set $\gamma = N$, so that $[\gamma] = \mathrm{PD}(w_1(\Sigma))$. In both cases, we have a decomposition

$$\Sigma = T(\gamma, \Sigma) \cup \Sigma'$$

where $\Sigma' = \Sigma \setminus \mathrm{Int}\, T(\gamma, \Sigma)$ is a compact connected orientable surface. Notice that $T(\gamma, \Sigma)$ is an annulus if $\chi(\Sigma)$ is even and it is a Möbius strip otherwise. Since $T(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^2) \subset \mathbb{R}\mathrm{P}^2$ is a Möbius strip, there is a d-fold cyclic unbranched covering

$$p_1 : T(\gamma, \Sigma) \xrightarrow{d:1} T(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^2)$$

for every integer $d \geqslant 2$ with the same parity of $\chi(\Sigma)$.

We want to extend such covering to a d-fold simple branched covering $\Sigma \to \mathbb{R}\mathrm{P}^2$. In order to do so, we first observe that the orientable surface Σ' has one or two boundary components depending on whether d is odd or even. In both cases, a 2-fold simple branched covering $\Sigma' \to D^2$ can be obtained as a suitable restriction of a hyper-elliptic covering of S^2. By stabilizing such covering $d-2$ times, we get a d-fold simple branched covering

$$p_2 : \Sigma' \xrightarrow{d:1} D^2.$$

In particular, we choose to perform the stabilizations in such a way that the monodromy $\omega_{p_2}([\partial D^2])$ is a cycle of length d if d is odd, while it is a product of two disjoint cycles of length $d/2$ if d is even. Then, the d-fold simple branched coverings p_1 and p_2 can be glued together along the boundary to get a d-fold simple branched covering

$$p = p_1 \cup_\partial p_2 : \Sigma \xrightarrow{d:1} \mathbb{R}\mathrm{P}^2.$$

The gluing compatibility can be checked by arguing that the monodromy $\omega_{p_1}([\mathbb{R}\mathrm{P}^1])$ is a cycle of length d, whose square is again a cycle of length d if d is odd, while it is a product of two disjoint cycles of length $d/2$ if d is even. □

The next lemma will be our main tool for constructing 4-dimensional branched coverings of the tubular neighbourhood of $\mathbb{R}\mathrm{P}^3$ inside $\mathbb{R}\mathrm{P}^4$ as fiberwise extensions of 3-dimensional branched coverings.

Lemma 1.8. *Given a closed connected orientable 3-manifold Y, let $c \in H^1(Y; \mathbb{Z}_2)$ be any cohomology class. Then, for every $d \geqslant 3$ such that $d \equiv \langle c^3, [Y]\rangle \pmod 2$, there is a d-fold simple covering*

$$q : Y \xrightarrow{d:1} \mathbb{R}\mathrm{P}^3$$

branched over a link, such that $c = q^*(c_0)$, where c_0 is the non-zero element of $H^1(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2)$.

Proof. Let $\Sigma \subset Y$ be a closed connected surface representing the Poincaré dual $\mathrm{PD}(c) \in H_2(Y; \mathbb{Z}_2)$. For every $d' \geqslant 3$ such that $d' \equiv \chi(\Sigma) \pmod 2$ we will now build a d'-fold simple branched covering $q : Y \to \mathbb{R}\mathrm{P}^3$ such that $q^*(c_0) = c$. Consider the decompositions

$$Y = T(\Sigma, Y) \cup Y' \quad \text{and} \quad \mathbb{R}\mathrm{P}^3 = T(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^3) \cup D^3$$

where $Y' = Y \setminus \operatorname{Int} T(\Sigma, Y)$. Lemma 1.7 implies the existence of a d'-fold simple branched covering $p : \Sigma \to \mathbb{RP}^2$. Since the ambient 3-manifold Y is orientable, $w_1(\Sigma) = w_1(\nu_Y \Sigma)$. Then, the pullback under p of ν_0, the normal D^1-bundle of $\mathbb{RP}^2$ in $\mathbb{RP}^3$, is ν, the normal D^1-bundle of Σ in Y, that is we have the following commutative diagram

$$\begin{array}{ccc} T(\Sigma, Y) & \xrightarrow{q_1} & T(\mathbb{RP}^2, \mathbb{RP}^3) \\ \nu \big\downarrow & & \big\downarrow \nu_0 \\ \Sigma & \xrightarrow[p]{} & \mathbb{RP}^2. \end{array}$$

Here the lifting q_1 of p is a d'-fold simple branched covering which fiberwise extends p and $B_{q_1} = \nu_0^{-1}(B_p)$. Its restriction to the boundary is a branched covering

$$q_1|_\partial : \partial T(\Sigma, Y) \xrightarrow{d' : 1} \partial T(\mathbb{RP}^2, \mathbb{RP}^3) = S^2.$$

Since the degree of $q_1|_\partial$ is $d' \geqslant 3$, by [2, Corollary 6.3] $q_1|_\partial$ extends to a d'-fold simple covering

$$q_2 : Y' \xrightarrow{d' : 1} D^3$$

branched over a proper 1-dimensional submanifold of D^3. Then, the conclusion follows by setting $q = q_1 \cup_\partial q_2$. Notice that $q^*(c_0) = c$ by construction and $B_q = B_{q_1} \cup B_{q_2}$ is a link in $\mathbb{RP}^3$. Moreover, the parity of d' must coincide with that of $d \equiv \langle c^3, [Y] \rangle \pmod 2$, since

$$d \equiv \langle c^3, [Y] \rangle = \langle q^*(c_0)^3, [Y] \rangle = d' \langle c_0^3, [\mathbb{RP}^3] \rangle \equiv d' \pmod 2.$$

Hence, we can take $d' = d$ in the above construction. □

As a straightforward consequence of Lemma 1.8, we get the following.

Lemma 1.9. *Let $\xi : T \to Y$ be a D^1-bundle over a closed connected orientable 3-manifold Y. For every $d \geqslant 3$ such that $d \equiv \langle w_1(\xi)^3, [Y] \rangle \pmod 2$, there is a commutative diagram*

$$\begin{array}{ccc} T & \xrightarrow{\widehat{q}} & T(\mathbb{RP}^3, \mathbb{RP}^4) \\ \xi \big\downarrow & & \big\downarrow \xi_0 \\ Y & \xrightarrow[q]{} & \mathbb{RP}^3 \end{array}$$

where q and $\widehat{q}$ are d-fold simple branched coverings and ξ_0 is the projection map of the tubular neighbourhood of $\mathbb{RP}^3 \subset \mathbb{RP}^4$. In particular, $B_q \subset \mathbb{RP}^3$ is a link and $B_{\widehat{q}} = \xi_0^{-1}(B_q)$ is a non-singular proper surface.

Proof. Apply Lemma 1.8 to get a d-fold simple branched covering $q : Y \to \mathbb{R}\mathrm{P}^3$ such that $q^*(c_0) = w_1(\xi)$. Then, the pullback of ξ_0 with respect to q is a D^1-bundle over Y isomorphic to ξ. We can hence define $\widehat{q} : T \to T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4)$ to be the fiberwise extension of q (or, equivalently, its pullback under ξ_0), which satisfies the required conditions. □

Fillability properties of branched coverings of S^3

The following fillability notion will be crucial for the construction of the branched coverings in the proof of Theorems A and B.

Definition 1.10. A d-fold simple branched covering $p : Y \to S^3$ is said to be *ribbon fillable* if it can be extended to a d-fold simple branched covering $q : X \to D^4$ whose branch set $B_q \subset D^4$ is a properly embedded ribbon surface. This immediately implies that $\partial X = Y$ and that $\partial B_q = B_p$ as labeled links.

Remark 1.11. As observed in [11, Section 1], the above definition is invariant under equivalence of p up to homeomorphisms and this implies that the ribbon fillability of p can be expressed in terms of the labeled branch set B_p by requiring that it is a labeled link in S^3 bounding a labeled ribbon surface in D^4.

In the following, we will denote by Γ_d the cobordism group of d-fold simple coverings of S^3 branched over a link. More precisely, we set

$$\Gamma_d = \{p : Y \to S^3 \text{ a } d\text{-fold simple covering branched over a link}\}/\!\sim,$$

where Y is any closed oriented 3-manifold and $p_0 \sim p_1$ if and only if there is a d-fold simple covering $q : X \to S^3 \times I$ branched over a proper non-singular surface in $S^3 \times I$, which restricts to p_i over $S^3 \times \{i\}$ for $i = 0, 1$. A group operation can be defined on Γ_d by setting $[p] + [p'] = [p'']$, where p'' is the branched covering described by the separated union of the labeled branch links of p and p'.

The group Γ_d^{or} is defined analogously, by additionally requiring that the branch set of the covering p is an oriented link and the branch set of the cobordism q is an oriented surface.

It is worth noting that a d-fold simple covering $p : Y \xrightarrow{d:1} S^3$ branched over a link represents the trivial class in Γ_d if and only if B_p bounds a proper non-singular labeled surface $F \subset D^4$ (required to be orientable in the case of Γ_d^{or}), which represents a d-fold simple branched covering $\widetilde{p} : X \to D^4$ such that $\partial X = Y$, $\widetilde{p}|_Y = p$ and $\partial F = B_p$ as labeled links.

The relevance of ribbon fillability in our context is due to the following two results in [11]. In particular, Theorem 1.12 is a rephrasing of [11, Theorem 1.2], which is one of the main results therein, while Theorem 1.13 is stated without proof as [11, Theorem 1.8]. Since we will make use of the latter theorem in the present work, we provide a proof of it. We emphasise that, while in Theorem 1.12 the 4-manifold X is required to have exactly n boundary connected components, in Theorem 1.13 we allow this number to possibly be greater than n.

Theorem 1.12 (Piergallini–Zuddas)**.** *Let X be a compact connected oriented PL 4-manifold with $n \geqslant 1$ boundary components and let $p : \partial X \to \sqcup_n S^3$ be a disjoint separated union of d-fold ribbon fillable simple branched coverings, with $d \geqslant 4$. Then, p can be extended to a d-fold simple branched covering $q : X \to S^4 \setminus \sqcup_n \operatorname{Int} D^4$ whose branch set B_q satisfies one of the above conditions* (a) *or* (b)*.*

Theorem 1.13 (Piergallini–Zuddas)**.** *Let X be a compact connected oriented PL 4-manifold and let $p : \partial X \to \sqcup_n S^3$ be a disjoint union of d-fold ribbon fillable simple branched coverings, with $d \geqslant 4$ and $n \geqslant 1$. Then, p can be extended to a d-fold simple branched covering $q : X \to S^4 \setminus \sqcup_n \operatorname{Int} D^4$ whose branch set B_q satisfies one of the above conditions* (a) *or* (b)*.*

Proof. Let $Y_1, \dots, Y_m$ be the connected components of ∂X for a certain $m \geqslant n$. We prove the statement by induction on m starting from the case $m = n$, where the conclusion directly follows from Theorem 1.12.

Suppose now that $m > n$. Denote by $C \cong \sqcup_{i=1}^m Y_i \times [0,1]$ a collar of ∂X in X and by $C_i \cong Y_i \times [0,1]$ the connected component of C containing Y_i. We number the 3-spheres in the disjoint union as $S^3_1, \dots, S^3_n$ in such a way that over S^3_1 there are at least two components of ∂X. Accordingly, we set

$$p_i = p| : p^{-1}(S^3_i) \xrightarrow{d:1} S^3_i.$$

We assume that the components Y_i are numbered in such a way that $p^{-1}(S^3_1) = Y_1 \sqcup \cdots \sqcup Y_k$ for some $k \geqslant 2$. We thicken p_1 by crossing with the identity of $[0,1]$ to get a simple d-fold branched covering

$$q_1 = p_1 \times \mathrm{id}_{[0,1]} : \sqcup_{i=1}^k C_i \xrightarrow{d:1} S^3_1 \times [0,1].$$

We number the sheets of q_1 so that the first and the second sheet lie in the components C_1 and C_2 respectively. Moreover, since X is connected, we can find inside $X \setminus \operatorname{Int}(C)$ a 1-handle H^1 attached to C that joins C_1 and C_2. Then, we can extend q_1 over H^1 to get a simple d-fold branched covering

$$q_1' : \sqcup_{i=1}^k C_i \cup H^1 \xrightarrow{d:1} S^3_1 \times [0,1]$$

having as branch set the union of B_{q_1} and a separated proper trivial 2-disk with monodromy (1 2), whose boundary is contained in $S^3_1 \times \{1\}$. In other words, q'_1 is obtained by performing an equivariant boundary connected sum between q_1 and the d-fold branched covering $\sqcup_{d-1} D^4 \mapsto D^4$ whose branch set is a proper trivial 2-disk with monodromy (1 2). By construction, the restriction of q'_1 over $S^3_1 \times \{1\}$ is a ribbon fillable simple d-fold branched covering

$$p'_1 = q'_1|_{S^3_1 \times \{1\}} : (Y_1 \# Y_2) \sqcup Y_3 \sqcup \cdots \sqcup Y_k \xrightarrow{d:1} S^3_1.$$

We consider the union of d-fold ribbon fillable simple branched coverings

$$p' = p'_1 \sqcup p_2 \sqcup \cdots \sqcup p_n : (Y_1 \# Y_2) \sqcup Y_3 \sqcup \cdots \sqcup Y_m \xrightarrow{d:1} \sqcup_{i=1}^n S^3_i$$

having as domain a 3-manifold with $m-1$ connected components. By the inductive hypothesis, we can extend p' to a simple d-fold branched covering

$$q_2 : X \setminus \mathrm{Int}(\sqcup_{i=1}^k C_i \cup H^1) \xrightarrow{d:1} S^4 \setminus \mathrm{Int}(\sqcup_n D^4)$$

whose branch set satisfies one of the conditions (a) or (b). The conclusion follows by setting $q = q'_1 \cup_\partial q_2$. □

The next lemma tells us that a d-fold simple branched covering over S^3 is ribbon fillable if and only if it represents the null element in Γ_d.

Lemma 1.14. *Let $p : Y \to S^3$ be a connected d-fold simple branched covering whose labeled branch set $B_p \subset S^3$ bounds a proper non-singular surface $F \subset D^4$ labeled with transpositions in S_d. Then B_p also bounds a ribbon surface $F' \subset D^4$ which is again labeled with transpositions in S_d. Moreover, F' can be taken orientable if F is orientable. Therefore, p represents the null class in Γ_d (resp. Γ_d^{or}) if and only if p is ribbon fillable (resp. ribbon fillable by an orientable ribbon surface).*

Proof. Without loss of generality, we can assume that the restriction $\rho|_{\mathrm{Int}\, F}$ of the squared norm function $\rho := \|\cdot\|^2 : D^4 \to \mathbb{R}$ is Morse. Then F is ribbon if and only if $\rho|_{\mathrm{Int}\, F}$ has no local maxima.

If F is not ribbon, by general position we also assume that each radius of D^4 containing a local maximum of $\rho|_{\mathrm{Int}\, F}$ does not meet F elsewhere. Then, we proceed by subsequently eliminating the local maxima as follows. Let k be the number of the local maxima of $\rho|_{\mathrm{Int}\, F}$. Pick one of them and push a small 2-disk neighbourhood of it in F radially through S^3. In this way, we obtain a surface that intersects S^3 along B_p and a separated 2-disk. By removing the interior of that disk we get a proper non-singular labeled surface whose boundary is the union of B_p and a separated unknot labeled with a single transposition in

S_d. That unknot can be joined to B_p by a suitable labeled band in S^3 (this is the only point where we need that Y is connected and F is labeled with transpositions in S_d). Finally, by pushing the interior of the band inside D^4 we get a proper non-singular labeled surface, that we still denote by F, such that $\rho|_{\mathrm{Int}\,F}$ has $k-1$ local maxima and whose boundary is label-isotopic to B_p. By iterating this modification over all the local maxima, we eventually obtain the desired labeled ribbon surface F'.

Notice that this operation preserves the orientability of the labeled surfaces involved in the construction, although it changes its topology, since $\chi(F') = \chi(F) - 2k$, as well as the topology of the interior of the covering 4-manifold. □

We recall the following result from [5, Theorems 9 and 11].

Theorem 1.15 (Hilden–Little)**.** *There are exactly three cobordism classes of 3-fold simple coverings $p : Y \to S^3$ branched over oriented links, and hence Γ_3^{or} is cyclic of order three. A generator is given by the irregular 3-fold simple covering $S^3 \to S^3$ branched over an oriented (left or right-handed) trefoil knot, see Figure 1.*

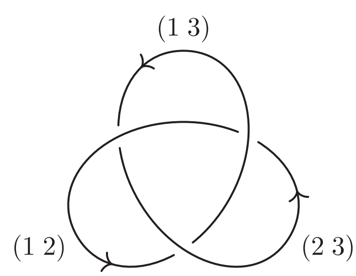

Figure 1. The labeled branch set of a generator of Γ_3^{or}.

The next lemma is a direct consequence of Theorem 1.15.

Lemma 1.16. *Applying an oriented Montesinos move $C^{\pm}$ (see Figure 2) to a 3-fold simple covering $p : Y \to S^3$ branched over an oriented link changes the cobordism class of the covering in $\Gamma_3^{\mathrm{or}} \cong \mathbb{Z}_3$ by ± 1, while leaving Y unchanged up to homeomorphism.*

Proof. That the moves $C^{\pm}$ do not change the manifold Y is a well known fact (cf. [7, 8]). For the rest of the statement it suffices to observe that, up to oriented cobordism, performing a $C^{\pm}$ move is the same as adding/deleting a separated copy of the labeled knot in Figure 1. □

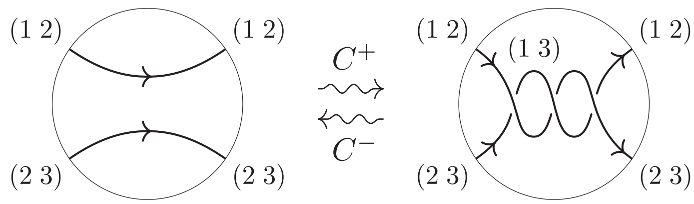

FIGURE 2. The oriented Montesinos moves $C^{\pm}$.

Remark 1.17. The oriented Montesinos moves $C^{\pm}$ can be applied, up to label conjugation in S_d, to any connected d-fold simple covering between 3-manifolds branched over a link, with $d \geqslant 3$. This operation does not change the homotopy class of the given branched covering, since the modifications occur locally inside 3-disks.

2. Proofs of the main results

This section is devoted to the proofs of our main results.

Proof of Theorem A

We start by proving the necessary condition on the degree. Given any d-fold branched covering $p : X \to \mathbb{R}\mathrm{P}^4$ we have

$$\begin{aligned}\langle w_1(X)^4, [X]\rangle &= \langle p^*(w_1(\mathbb{R}\mathrm{P}^4))^4, [X]\rangle = \langle w_1(\mathbb{R}\mathrm{P}^4)^4, p_*[X]\rangle \\ &= \langle w_1(\mathbb{R}\mathrm{P}^4)^4, d\,[\mathbb{R}\mathrm{P}^4]\rangle \equiv d \pmod 2.\end{aligned}$$

For the converse, let $d \geqslant 4$ such that $\langle w_1(X)^4, [X]\rangle \equiv d \pmod 2$ and consider the decompositions

$$X = T(Y, X) \cup_\partial X' \quad \text{and} \quad \mathbb{R}\mathrm{P}^4 = T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4) \cup_\partial D^4$$

given by Lemma 1.2, where $Y \subset X$ is a connected orientable 3-dimensional submanifold such that

$$[Y] = \mathrm{PD}(w_1(X)) \in H_3(X; \mathbb{Z}_2)$$

and $X' = X \setminus \operatorname{Int} T(Y, X)$. Let $\Sigma \subset Y$ be a connected surface such that

$$[\Sigma] = \mathrm{PD}(w_1(\nu_X Y)) \in H_2(Y; \mathbb{Z}_2).$$

Notice that Σ is obtained by intersecting Y with a transversal copy Y'. Then, we have

$$\begin{aligned}\langle w_1(X)^4, [X]\rangle &= [Y \cap Y']^2 = [\Sigma]^2 = \langle w_2(\nu_X \Sigma), [\Sigma]\rangle \\ &= \langle w_2(\nu_Y \Sigma \oplus \nu_X Y|_\Sigma), [\Sigma]\rangle = \langle w_1(\nu_Y \Sigma) \smile w_1(\nu_X Y|_\Sigma), [\Sigma]\rangle \\ &= \langle w_1(\nu_Y \Sigma)^2, [\Sigma]\rangle = \langle w_1(\Sigma)^2, [\Sigma]\rangle \equiv \chi(\Sigma) \pmod 2.\end{aligned}$$

Here, we use the isomorphism of line bundles $\nu_Y \Sigma \cong \nu_X Y|_\Sigma$ given by the fact that $\Sigma = Y \cap Y'$ and the equality $w_1(\Sigma) = w_1(\nu_Y \Sigma)$.

Also note that we have the congruence $d \equiv \langle w_1(\nu_X Y)^3, [Y]\rangle \pmod 2$, which is implied by the fact that

$$\langle w_1(\nu_X Y)^3, [Y]\rangle = \langle w_1(X)^3, i_*([Y])\rangle = \langle w_1(X)^4, [X]\rangle,$$

where i_* is the homomorphism induced in homology by the inclusion $i : Y \to X$. The first equality follows from the orientability of Y, while the second one is a consequence of the duality between the Kronecker product and the algebraic intersection of homology classes.

We will now construct a d-fold simple branched covering

$$p : X \xrightarrow{d:1} \mathbb{R}\mathrm{P}^4$$

for any $d \geqslant 4$ such that $d \equiv \langle w_1(X)^4, [X]\rangle \pmod 2$.

We distinguish two cases, according to the parity of d.

Case 1: *d* odd. By Lemma 1.9, there exists a 3-fold simple branched covering

$$q : Y \xrightarrow{3:1} \mathbb{R}\mathrm{P}^3,$$

which admits a fiberwise extension to a 3-fold simple branched covering

$$\widehat{q} : T(Y, X) \xrightarrow{3:1} T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4).$$

Notice that the restriction of $\widehat{q}$ over $\partial T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4)$ coincides with the pullback

$$\widetilde{q} : \partial T(Y, X) \xrightarrow{3:1} \partial T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4)$$

of q under the universal covering $u : S^3 \to \mathbb{R}\mathrm{P}^3$, which is given by the restriction of the projection $T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4) \to \mathbb{R}\mathrm{P}^3$ to the boundary.

We can suppose without loss of generality that $\widetilde{q}$ is ribbon fillable. Indeed, if this is not the case, Lemma 1.14 implies that $\widetilde{q}$ does not represent the trivial element in the cobordism group Γ_3^{or} of 3-fold simple branched coverings for any orientation of the branch set $B_{\widetilde{q}}$. Fixed an arbitrary orientation on B_q, we consider the u-invariant orientation induced on $B_{\widetilde{q}}$ and observe that applying one oriented Montesinos move $C^\pm$ to q amounts to applying two oriented Montesinos moves of the same type to the pullback $\widetilde{q}$. Then, by Theorem 1.15 and Lemma 1.16 a single oriented Montesinos move on q suffices to make $\widetilde{q}$ trivial in $\Gamma_3^{\mathrm{or}} \cong \mathbb{Z}_3$, and hence ribbon fillable by Lemma 1.14. Notice that the domains of $\widehat{q}$ and $\widetilde{q}$ are unchanged, since in virtue of Remark 1.17 an oriented Montesinos move $C^\pm$ does not change the homotopy class of the maps and hence they induce the same pullbacks.

At this point, we assume that $\widetilde{q}$ is ribbon fillable. We stabilize k times q as in Proposition 1.5 to get a d-fold simple branched covering

$$q' : Y \xrightarrow{d:1} \mathbb{R}\mathrm{P}^3$$

with $k = (d-3)/2$. Then, we consider its fiberwise extension

$$\widehat{q}' : T(Y, X) \xrightarrow{d:1} T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4),$$

whose total space is still $T(Y, X)$, since thanks to Remark 1.6 we have

$$q^*(w_1(\nu_{\mathbb{R}\mathrm{P}^4}\mathbb{R}\mathrm{P}^3)) = (q')^*(w_1(\nu_{\mathbb{R}\mathrm{P}^4}\mathbb{R}\mathrm{P}^3)),$$

and the D^1-bundles associated to $q^*(\nu_{\mathbb{R}\mathrm{P}^4}\mathbb{R}\mathrm{P}^3)$ and $(q')^*(\nu_{\mathbb{R}\mathrm{P}^4}\mathbb{R}\mathrm{P}^3)$ are hence isomorphic. We also consider the restriction of $\widehat{q}'$ to the boundary

$$\widetilde{q}' : \partial T(Y, X) \xrightarrow{d:1} \partial T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4).$$

The branch set of q' is the union of B_q and k separated unknots with monodromies $(3\ 4), (5\ 6), \dots, (d{-}2\ d{-}1)$ and thus $\widetilde{q}'$ is still ribbon fillable, given that its branch set is the union of $B_{\widetilde{q}}$ with $d-3$ separated unknots (in fact, it is obtained from $\widetilde{q}$ by stabilizing $d-3$ times). We can hence apply Theorem 1.12 with $n = 1$ to extend $\widetilde{q}'$ and hence $\widehat{q}'$ as well to a d-fold simple branched covering

$$p : X = T(Y, X) \cup_\partial X' \xrightarrow{d:1} \mathbb{R}\mathrm{P}^4 = T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4) \cup_\partial D^4$$

whose branch set B_p is a non-singular locally flat surface, satisfying condition (b).

Case 2: *d* even. By Lemma 1.7 and its proof, there is a 2-fold simple branched covering

$$\Sigma \xrightarrow{2:1} \mathbb{R}\mathrm{P}^2$$

such that the standard $\mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$ is disjoint from the branch set and its monodromy is the transposition $(1\ 2)$. By the Riemann–Hurwitz formula, the number of branch points is $h = 2 - \chi(\Sigma)$, which is even by the assumption on the parity of d. Since $w_1(\Sigma) = w_1(\nu_Y\Sigma)$ and $w_1(\mathbb{R}\mathrm{P}^2)$ pulls back to $w_1(\Sigma)$, we can extend this covering fiberwise to a 2-fold simple branched covering

$$T(\Sigma, Y) \xrightarrow{2:1} T(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^3).$$

Then, we stabilize the latter map as in Lemma 1.4 to get a 4-fold simple branched covering

$$q_1 : T(\Sigma, Y) \setminus \mathrm{Int}\, D^3 \xrightarrow{4:1} T(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^3),$$

whose branch set consists of the preimages with respect to the projection $T(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^3) \to \mathbb{R}\mathrm{P}^2$ of the h branch points in $\mathbb{R}\mathrm{P}^2$ and a proper separated trivial arc with monodromy $(2\ 3)$. Moreover, after such stabilization, the monodromy of the standard $\mathbb{R}\mathrm{P}^1$ is given by

$$\omega_{q_1}([\mathbb{R}\mathrm{P}^1]) = (1\ 2)(3\ 4).$$

Notice that the restriction of q_1 to the boundary is a 4-fold simple branched covering

$$q_1|_\partial : \partial T(\Sigma, Y) \sqcup S^2 \xrightarrow{4:1} S^2$$

which is 3-fold on $\partial T(\Sigma, Y)$ and one to one on S^2. We can extend this restriction to a 4-fold simple branched covering

$$q_2 : (Y \setminus \operatorname{Int} T(\Sigma, Y)) \sqcup D^3 \xrightarrow{4:1} D^3$$

by applying the Berstein–Edmonds extension result [2, Corollary 6.3] on $Y \setminus \operatorname{Int} T(\Sigma, Y)$ and taking the disjoint union with the identity map of D^3. We can then glue q_1 and q_2 along the boundary to get a 4-fold simple branched covering

$$q = q_1 \cup_\partial q_2 : Y \xrightarrow{4:1} \mathbb{R}\mathrm{P}^3.$$

Since $[\Sigma] = [q^{-1}(\mathbb{R}\mathrm{P}^2)]$ by construction, it follows by Poincaré duality that $w_1(\nu_X Y) = q^*(w_1(\nu_{\mathbb{R}\mathrm{P}^4} \mathbb{R}\mathrm{P}^3))$. Hence, we can extend q fiberwise to a 4-fold simple branched covering

$$\widehat{q} : T(Y, X) \xrightarrow{4:1} T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4),$$

whose restriction

$$\widetilde{q} : \partial T(Y, X) \xrightarrow{4:1} \partial T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4) = S^3$$

can be identified with the pullback of q under the universal covering $u : S^3 \to \mathbb{R}\mathrm{P}^3$.

Claim. *The branched covering $\widetilde{q}$ can be assumed to be ribbon fillable, up to modifying q by a suitable Montesinos move $C^\pm$. In particular, the domains of $\widehat{q}$ and $\widetilde{q}$ are unchanged, as in Case 1.*

Proof of the claim. Notice that the branch set of $\widetilde{q}$ is symmetric with respect to the antipodal map $\alpha : S^3 \to S^3$, which acts on the set of labels by conjugation with $(1\ 2)(3\ 4)$. In particular, the equatorial S^2 in S^3 given by the preimage of the standard $\mathbb{R}\mathrm{P}^2 \subset \mathbb{R}\mathrm{P}^3$ intersects the branch set of $\widetilde{q}$ in exactly $2h$ points with monodromy $(1\ 2)$. Moreover, such S^2 determines a splitting

$$S^3 = D^3_1 \cup_{S^2} D^3_2$$

into two half 3-spheres. We also consider the decomposition

$$D^4 = D^4_1 \cup_{D^3_0} D^4_2,$$

where $D^3_0 \subset D^4$ is the standard 3-disk spanned by $S^2 \subset S^3$ and the D^4_i's are two half 4-disks, whose boundaries will be denoted by

$$S^3_i = \partial D^4_i = D^3_0 \cup_{S^2} D^3_i$$

for $i = 1, 2$. By construction, $\widetilde{q}$ has a 3-fold connected component and a separated trivial sheet over each of the 3-disks D^3_1 and D^3_2, and the labels of $B_{\widetilde{q}}$ can be $(1\ 2), (1\ 3), (2\ 3)$ in D^3_1 and $(1\ 2), (1\ 4), (2\ 4)$ in D^3_2. This follows from the construction of q and by noticing that up to conjugation with $(1\ 2)(3\ 4)$ we have that $(1\ 2)$ corresponds to itself, $(2\ 3)$ corresponds to $(1\ 4)$ and $(1\ 3)$ corresponds to $(2\ 4)$.

A labeled ribbon surface in D^4 bounding $B_{\widetilde{q}}$ can be constructed as follows. Join two by two the $2h$ branch points in $B_{\widetilde{q}} \cap S^2$ with an α-invariant collection of disjoint arcs $\gamma_1, \dots, \gamma_h \subset S^2$. These can be constructed by taking the preimage of disjoint arcs in $\mathbb{RP}^2$ connecting in pairs the (even number of) branch points of the 2-fold cover $\Sigma \to \mathbb{RP}^2$.

For $i = 1, 2$, we define the labeled link

$$L_i = (B_{\widetilde{q}} \cap D^3_i) \cup \gamma_1 \cup \dots \cup \gamma_h \subset S^3_i,$$

where the arcs $\gamma_1, \dots, \gamma_h$ have the label $(1\ 2)$, so that L_i represents a connected 3-fold simple branched covering of S^3_i. Up to performing a single oriented Montesinos move $C^{\pm}$ on q as in Case 1, we can assume that L_1 bounds an oriented labeled ribbon surface $R_1 \subset D^4_1$, thanks to Lemma 1.16 and Lemma 1.14. By forgetting the orientation and acting on R_1 with the antipodal map of S^3 we get a ribbon surface $R_2 \subset D^3_2$ for L_2, whose labels are obtained from those of R_1 by conjugation with $(1\ 2)(3\ 4)$. Now, we have that

$$R_1 \cap R_2 = \partial R_1 \cap \partial R_2 = \gamma_1 \cup \dots \cup \gamma_h$$

with labels $(1\ 2)$. Therefore, $R := R_1 \cup R_2 \subset D^4$ is a labeled ribbon surface having $B_{\widetilde{q}}$ as the boundary. □

At this point, we resume the proof of Theorem A with $\widetilde{q}$ ribbon fillable. We can stabilize $k = (d-4)/2$ times q as in Proposition 1.5 to get a d-fold simple branched covering

$$q' : Y \xrightarrow{d:1} \mathbb{RP}^3,$$

whose fiberwise extension

$$\widehat{q}' : T(Y, X) \xrightarrow{d:1} T(\mathbb{RP}^3, \mathbb{RP}^4),$$

is still defined on $T(Y, X)$ in virtue of Remark 1.6, as in Case 1. Moreover, we denote by

$$\widetilde{q}' : \partial T(Y, X) \xrightarrow{d:1} \partial T(\mathbb{RP}^3, \mathbb{RP}^4)$$

the restriction of $\widehat{q}'$ to the boundary. As in the previous case, $\widetilde{q}'$ is still ribbon fillable, since its branch set is the union of $B_{\widetilde{q}}$ with $d-4$ separated unknots (in fact, it is obtained from $\widetilde{q}$ by stabilizing $d-4$ times).

Theorem 1.13 with $n = 1$ implies that we can extend $\widetilde{q}'$ and hence $\widehat{q}'$ as well to a d-fold simple branched covering

$$p : X = T(Y, X) \cup_\partial X' \xrightarrow{d:1} \mathbb{R}\mathrm{P}^4 = T(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^4) \cup_\partial D^4$$

whose branch set $B_p \subset \mathbb{R}\mathrm{P}^4$ has the desired properties (a) or (b). □

Proof of Theorem B

We start by proving the necessary condition. Let $\widetilde{w}_1(S^1 \widetilde{\times} S^3)$ be any integral lift of $w_1(S^1 \widetilde{\times} S^3)$. If there is a d-fold branched covering $p : X \to S^1 \widetilde{\times} S^3$, then $p^*(\widetilde{w}_1(S^1 \widetilde{\times} S^3))$ is an integral lift of $w_1(X)$.

Conversely, let $d \geqslant 4$ be fixed and suppose that $w_1(X)$ admits an integral lift. Lemma 1.2 implies the existence of a decomposition

$$X = T(Y, X) \cup_\partial X',$$

where $Y \subset X$ is a connected orientable 3-manifold (denoted by N in the lemma) having trivial tubular neighbourhood and whose $\mathbb{Z}_2$-homology class satisfies $[Y] = \mathrm{PD}(w_1(X))$, while $X' = X \setminus \mathrm{Int}\, T(Y, X)$. As the first step in the construction of the wanted branched covering $p : X \to S^1 \widetilde{\times} S^3$, we consider a d-fold ribbon fillable simple branched covering

$$q : Y \xrightarrow{d:1} S^3.$$

This can be obtained by stabilizing $d - 3$ times a 3-fold ribbon fillable simple branched covering $Y \to S^3$, whose existence is established by Montesinos in [6] or by Bobtcheva–Piergallini in [3]. Since $T(Y, X)$ is a product neighbourhood, we can extend q to a simple d-fold covering

$$\widehat{q} \cong q \times \mathrm{id}_{D^1} : T(Y, X) \cong Y \times D^1 \xrightarrow{d:1} T(S^3, S^1 \widetilde{\times} S^3) \cong S^3 \times D^1$$

branched over a collection of proper annuli, where S^3 is a fiber of the twisted bundle $S^1 \widetilde{\times} S^3$. This restricts to the boundary as a d-fold simple branched covering

$$\widetilde{q} \cong \sqcup_2\, q : \partial T(Y, X) \cong \sqcup_2\, Y \xrightarrow{d:1} \partial T(S^3, S^1 \widetilde{\times} S^3) \cong \sqcup_2\, S^3,$$

whose restriction over each copy of S^3 is ribbon fillable. Since there is a homeomorphism

$$(S^1 \widetilde{\times} S^3) \setminus \mathrm{Int}(T(S^3, S^1 \widetilde{\times} S^3)) \cong S^3 \times D^1 \cong S^4 \setminus \mathrm{Int}(\sqcup_2\, D^4),$$

we can apply Theorem 1.12 with $n = 2$ to extend $\widetilde{q}$ and hence $\widehat{q}$ to a simple d-fold branched covering

$$p : X = T(Y, X) \cup_\partial X' \xrightarrow{d:1} S^1 \widetilde{\times} S^3 \cong T(S^3, S^1 \widetilde{\times} S^3) \cup_\partial (S^3 \times D^1)$$

satisfying the condition (a) or (b). Here, the gluing map of the last boundary union is the disjoint union of the identity map and a reflection of S^3. □

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Scuola Internazionale Superiore di Studi Avanzati (SISSA), Via Bonomea 265, 34136, Trieste, Italy.
Email address: vbais@sissa.it

Scuola di Scienze e Tecnologie, Università di Camerino, Via Madonna delle Carceri 9, 62032 Camerino, Italy.
Email address: riccardo.piergallini@unicam.it

Dipartimento di Matematica, Informatica e Geoscienze, Università degli Studi di Trieste, Via Valerio 12/1, 34127, Trieste, Italy.
Email address: dzuddas@units.it