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BRAIDING NON-ORIENTABLE SURFACES IN S^4

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Abstract

Closed braided surfaces in S^4 are the two-dimensional analogous of closed braids in S^3. They are useful in studying smooth closed orientable surfaces in S^4, since any such a surface is isotopic to a braided one. We show that the non-orientable version of this result does not hold, that is smooth closed non-orientable surfaces cannot be braided. In fact, any reasonable definition of non-orientable braided surfaces leads to very strong restrictions in terms of self-intersection and Euler characteristic.

Key words and phrases: non-orientable surfaces, braided surfaces in fibre bundles.
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Introduction

The concept of braided surface in $B^4 \cong B^2 \times B^2$ has been introduced since the early eighties by Rudolph (cf. [18], [19] and [20]) as a two dimensional analogous of the classical Artin's braids. Namely, he called braided a surface in $B^2 \times B^2$ which projects onto the first factor by a branched covering.

Successively, in the nineties, Viro and Kamada (cf. [8], [9] and [11]) considered closed braided surfaces in S^4, that is surfaces contained in a normal neighborhood of $S^2 \subset S^4$, projecting onto S^2 by a branched covering. We can think of a closed braided surface as closure of a Rudolph's braided surface with trivial boundary, just in the same way we think of a closed braid in S^3 as closure of an Artin's braid.

By Kamada's results, closed braided surfaces can be used to study orientable smooth surfaces in S^4. In fact, he provided two dimensional versions of the Alexander's and Markov's theorems on braids, by proving that any such a surface is isotopically equivalent to a closed braided surface and finding a set of moves relating isotopic braided surfaces.

In this paper we deal with the following question: can the above mentioned results be adapted in order to handle non-orientable surfaces in S^4, replacing the standard 2-sphere as base model for braided surfaces with some standard non-orientable surface, such as the Veronese surface (see section 3)?

The question is relevant in relation to the representation of orientable closed smooth 4-manifolds as branched covers of S^4, in which non-orientable surfaces play an essential role as branch sets in S^4 (see [6] and [17]).

Unfortunately, the answer is generally negative, in spite of some partial result obtained by Kamada (cf. [7]). In fact, in section 3 we show that there are very restrictive conditions for a non-orientable smooth surface in S^4 to be isotopic to a braided one. Nevertheless, we don't know whether any orientable smooth closed 4-manifold is a cover of S^4 branched over a (possibly non-orientable) braided surface.

In order to study non-orientable braided surfaces in S^4, in section 2 we consider braided surfaces in R^2-bundles over surfaces and prove a few of preliminary results about them, which are of some interest independently of the present application.

This paper is a revised version of part of the degree thesis [21] written by the second author under the supervision of the first author.

1. Preliminaries

To begin with, we reformulate in terms of coverings the classical Artin's notion of braid. By a *geometric braid* of degree d in R^3 we mean a 1-submanifold $b \subset [0,1] \times R^2 \subset R^3$ such that the canonical projection $\pi : [0,1] \times R^2 \to [0,1]$ restricts to a covering $\pi_{|b} : b \to [0,1]$ of degree d and moreover, putting $b_t = \{x \in R^2 \mid (t,x) \in b\}$, we have $b_0 = b_1 = *$ for a fixed $* = \{*_1, \dots, *_d\} \subset R^2$.

Considering braids of degree d up to fibre preserving (with respect to π) ambient isotopy of $[0,1] \times R^2$, we can think of them as elements of the *braid group* $B_d = \pi_1(S_d R^2, *)$ of degree d, where $S_d R^2 \cong (\Pi_d R^2 - \Delta)/\Sigma_d$ denotes the space of all the subsets of R^2 consisting of d distinct points.

We recall the standard presentation of B_d (cf. [4]), with generators $x_1, \dots, x_{d-1}$, defined as shown in figure 1, and relations $x_i x_j = x_j x_i$ for any $i, j = 1, \dots, d-1$ such that $|i-j| > 1$ and $x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$ for any $i = 1, \dots, d-2$.

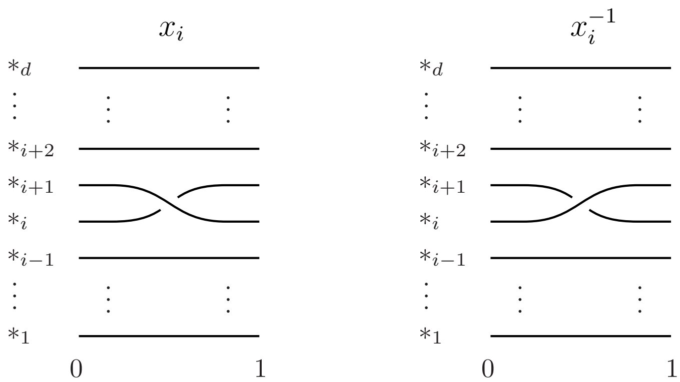

FIGURE 1.

Given a braid $b = x_{j_1}^{\varepsilon_1} \dots x_{j_k}^{\varepsilon_k} \in B_d$, we define the *index* of b to be the exponent sum $i(b) = \varepsilon_1 + \dots + \varepsilon_k$. Since all the relations above are balanced, it immediately follows that $i(b)$ is well-defined, that is it does not depend on the particular expression of b as a power product of standard generators.

We call a *closed braid* of degree d in R^3 any link $l \subset N(S^1) \subset R^3$, where $N(S^1)$ is a fixed open tubular neighborhood of S^1 in R^3, such that the ortogonal projection $\pi : N(S^1) \cong S^1 \times R^2 \to S^1$ restricts to a covering $\pi_{|l} : l \to S^1$ of degree d. By Alexander's theorem, any link in R^3 is ambient isotopic to a closed braid.

The *closure* of a braid $b \in B_d$ is the closed braid $\widehat{b} = (\varphi \times \mathrm{id}_{R^2})(b)$, where $\varphi : [0,1] \to S^1$ is the usual parametrization given by $\varphi(t) = (\cos 2\pi t, \sin 2\pi t)$. Of course, $\widehat{b}$ is defined only up to fibre preserving ambient isotopy of $N(S^1) \cong S^1 \times R^2$, being $b \in B_d$ defined only up to fiber preserving ambient isotopy of $[0,1] \times R^2$. Viceversa, $\widehat{b}$ uniquely determines b up to conjugation in B_d.

Then, it makes sense to define the index of a closed braid l in R^3 by putting $i(l) = i(b)$, where $b \in B_d$ is any braid such that $l = \widehat{b}$, being the index of braids obviously invariant under conjugation in B_d.

The index $i(l)$ of a closed braid l of degree d satisfies the following *Bennequin inequality* (cf. [3]), involving the Euler characteristic $\chi(S)$ of any surface $S \subset R^3$ such that $l = \operatorname{Bd} S$ (that is a Seifert surface for l): $|i(l)| \leq d - \chi(S)$.

Finally, we recall the notion of branched covering between surfaces, which is needed in order to consider braided surfaces. A map $p : S \to X$ between compact surfaces is called a *branched covering* iff at any $s \in S$ it is locally equivalent to the complex map $z \mapsto z^{d(s)}$, where $d(s) \geq 1$ is the *local degree* of p at s. The *branch points* of p are the images of the singular ones, that is of the points $s \in S$ such that $d(s) > 1$. Moreover, p a called *simple* if $d(s) = 2$ for any singular point $s \in S$ and p is injective on the singular points.

2. Braided surfaces in fiber bundles

Let $f : N \to X$ be an R^2-bundle over a compact connected surface X with (possibly empty) boundary. We call (*simple*) *braided surface* of degree d over X any locally flat compact surface $S \subset N$ such that the restriction $p = f_{|S} : S \to X$ is a (simple) branched covering of degree d. Moreover, we call *twist point* of S any singular point $t \in S$ of p and denote by $d(t) \geq 2$ the *local degree* of S at t, that is the local degree of p at t.

For any twist point $t \in S$, there exists a commutative diagram like the following, where: $C \subset N$ is a closed neighborhood of t, $D \subset X$ is a closed neighborhood of $p(t)$, h and k are homeomorphisms, $b_t \subset S^1 \times \operatorname{Int} B^2$ is a closed braid of degree $d(t)$, $C(b_t) \subset B^2 \times B^2$ is the cone of b_t with vertex $(0,0)$, π is the canonical projection on the first factor.

$$\begin{array}{ccccccc}
t & \in & S \cap C & \subset & C & \xrightarrow{f_{|C}} & D \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle k} \\
(0,0) & \in & C(b_t) & \subset & B^2 \times B^2 & \xrightarrow{\pi} & B^2
\end{array}$$

If N is oriented, we can assume that h is orientation preserving (with respect to the standard orientation of $B^2 \times B^2$). Moreover, fixed any local orientation of X at $p(t)$, we can also assume that k is orientation preserving (with respect to the standard orientation of B^2). With these two assumptions, b_t turns out to be uniquely determined up to braid isotopy, in such a way that we can define the *local index* $i(t)$ of S at t to be the integer number $i(b_t)$. In fact, it can be easily seen that $i(t)$ depends only on S and on the orientation of N, while it does not depend on the choice of the local orientation of X.

If t is a simple twist point of S, then, by local flatness, b_t coincides with the closure of one of the braids $x_1^{\pm 1} \in B_2$, so that $C(b_t)$ can be thought to have equation $w = z^2$ or $w = \bar{z}^2$ (depending on the sign of the exponent), with respect to the complex coordinates (w, z) of $B^2 \times B^2 \subset C^2$.

On the other hand, if t is a smooth twist point of S, then we get for $C(b_t)$ the equation $w = z^{d(t)}$ or $w = \bar{z}^{d(t)}$, while b_t turns out to be the closure of the braid $(x_1 \cdots x_{d(t)-1})^{\pm 1} \in B_{d(t)}$.

Hence, any simple twist point of S is smooth (up to fiber preserving ambient isotopy of N), and any smooth twist point t can be easily perturbed to get $d(t) - 1$ simple ones (up to ambient isotopy of N which does not preserve the fibers of f). In this case, we can assign to the twist point $t \in S$ a sign $s(t) = \pm 1$, depending only on the local shape of S and on the orientation of N, in such a way that $i(t) = s(t)(d(t) - 1)$.

For a non-smooth twist point t it may not exist any simple perturbation up to ambient isotopy, as it is shown in [10]. Nevertheless, we can modify S in a neighborhood of t in order to get a new braided surface S', where the twist point t is replaced by a certain number of simple twist points $t_1, \ldots, t_k$ such that $i(t) = i(t_1) + \ldots + i(t_k)$. Namely, if b_t is the closure of a braid $x_{j_1}^{\varepsilon_1} \cdots x_{j_k}^{\varepsilon_k} \in B_{d(t)}$, then we can replace $C(b_t)$ by a braided surface in $B^2 \times B^2$, having a positive (resp. negative) simple twist point t_l for each $\varepsilon_l = +1$ (resp. $\varepsilon_l = -1$), with $l = 1, \ldots, k$ (cf. proposition 1.11 of [19]).

We remark that the braided surface we put in place of $C(b_t)$ is not necessarily a disk, however it is homologous to $C(b_t)$ mod the common boundary b_t. Then, S and S' may not be isotopic, but they share some homological properties, in particular they have the same self-intersection number as multi-valued sections of f. Moreover, by local flatness, the closed braid b_t represents the unknot, hence the Bennequin inequality implies that $|i(t)| \leq d(t) - 1$.

Still assuming N oriented, we define the *index* of S as the sum $i(S) = \sum i(t)$, where t runs over all the twist points of S. The following proposition gives the index of S in terms of its degree d and of the Euler number e of the bundle $f : N \to X$ over a closed surface X.

PROPOSITION 2.1. *If N is oriented and $S \subset N$ is any braided surface over a closed surface X as above, then $i(S) = ed(d - 1)$.*

Proof. By the definition of index and the previous observations, we can assume that S is simple, so that $i(S)$ equals the algebraic number of the twist points of S.

To begin with, we consider a handlebody decomposition of the base surface X, consisting of one 0-handle H^0, 1-handles $H_1^1, \ldots, H_{2g+h}^1$ attached to H^0 in the standard way depicted in figure 2, with $g, h \geq 0$ and H_j^1 orientable (resp. non-orientable) for $j = 1, \ldots, 2g$ (resp. $j = 2g + 1, \ldots, 2g + h$), and one 2-handle H^2.

We can assume that all the branch points of p belong to $\operatorname{Int} H^2$, in such a way that, putting $X_1 = H^0 \cup H_1^1 \cup \ldots \cup H_{2g+h}^1$, $N_1 = f^{-1}(X_1)$ and $S_1 = S \cap N_1$, the restriction $p_{|S_1} : S_1 \to X_1$ is an ordinary covering.

By a suitable choice of the trivializations of f over the handles H^0 and H_j^1, we can think of N_1 as the quotient space obtained by attaching $H_j^1 \times R^2$ to $H^0 \times R^2$ for all $j = 1, \ldots, 2g + h$, by fiber preserving maps, whose restrictions to the fibers coincide with id_{R^2} or σ, where $\sigma : R^2 \to R^2$ is the symmetry with respect to the y-axis.

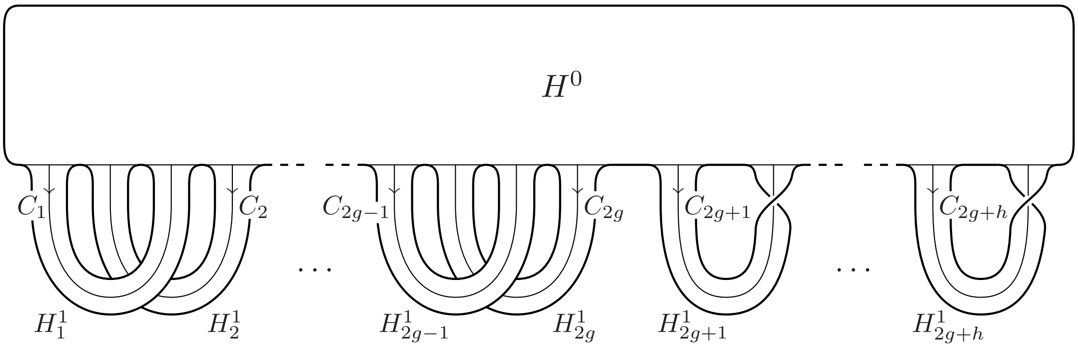

FIGURE 2.

Moreover, we can assume that the trivialization $f^{-1}(H^0) \cong H^0 \times R^2$, makes $S_0 = S \cap f^{-1}(H^0)$ into $H^0 \times \{*_1, \dots, *_d\} \subset H^0 \times R^2$, where $*_1, \dots, *_d$ belong to the x-axis and $\sigma(\{*_1, \dots, *_d\}) = \{*_1, \dots, *_d\}$. Then, for every $j = 1, \dots, 2g+h$, the trivialization $f^{-1}(H^1_j) \cong H^1_j \times R^2$ makes $S \cap f^{-1}(C_j)$ into a braid $c_j \subset C_j \times R^2$, where C_j denote the core of the handle H_j, oriented as in figure 2.

We observe that S_1 is completely determined (up to fiber preserving isotopy) by the braids $c_1, \dots, c_{2g+h} \in B_d$ and $\mathrm{Bd}\, S_1$ is a closed braid in $\mathrm{Bd}\, N_1 \cong \mathrm{Bd}\, X_1 \times R^2 \cong S^1 \times R^2$, which can be thought as the closure of the braid $c = c_1 c_2 c_1^{-1} c_2^{-1} \cdots c_{2g-1} c_{2g} c_{2g-1}^{-1} c_{2g}^{-1} c_{2g+1} c_{2g+1}^{\sigma} \cdots c_{2g+h} c_{2g+h}^{\sigma} \in B_d$, where c_j^{σ} denotes the image of c_j under the action of σ.

Putting $N_2 = f^{-1}(H^2)$ and $S_2 = S \cap N_2$, we have that $\mathrm{Bd}(S_2) = \mathrm{Bd}(S_1)$ is a closed braid in $\mathrm{Bd}\, N_2 \cong \mathrm{Bd}\, H^2 \times R^2 \cong S^1 \times R^2$, which can be thought as the closure of the braid $c' = ct^e$, where $t \in B_d$ denotes one positive full twist of d strings. On the other hand, denoting by $t_1, \dots, t_n \in S_2$ the (simple) twist points of S, it is straightforward (for example, see proposition 1.11 of [19]) to get $c' = y_1 x_{j_1}^{s(t_1)} y_1^{-1} \cdots y_n x_{j_n}^{s(t_n)} y_n^{-1}$, where each x_j is a standard generator of B_d and $s(t_j) = \pm 1$ as above.

At this point, we can finish the proof by observing that the computation of $i(c')$ based on the first expression of c' as a product of powers of generators gives us $ed(d-1)$, while the second one gives us $s(t_1) + \ldots + s(t_k)$, that is $i(S)$. $\square$

As a consequence of proposition 2.1, we get numerical obstructions to the existence of braided surfaces, in terms of Euler characteristic and number of twist points.

PROPOSITION 2.2. *If N is oriented and $S \subset N$ is any braided surface over a closed surface X as above, then $\chi(S) \leq d(\chi(X) - |e|(d-1))$. Moreover, if S is simple, then the number of twist points is even and not less than $|e|d(d-1)$.*

Proof. By the Hurwitz formula we have $\chi(S) = d\chi(X) - \sum(d(t)-1)$. Then, the first part of the proposition follows immediately by proposition 2.1 and by the inequalities $|i(S)| \leq \sum |i(t)| \leq \sum(d(t)-1)$. For the second part, it is enough to observe that, if S is simple, then the number of twist points coincides with $\sum(d(t)-1)$, which is congruent to $i(S) = \sum s(t)(d(t)-1) \bmod 2$. $\square$

Now, we want to show that the inequalities given by the proposition 2.2 are sharp. Given any R^2-bundle $f : N \to X$ with oriented total space N and arbitrary

Euler number $e > 0$ (the case $e < 0$ can be covered by reversing the orientation of N), let $S_1, \dots, S_d \subset N$ be d smooth sections of f, transversally meeting each other in e points. Then, replacing each of the $ed(d-1)/2$ double points of $S_1 \cup \dots \cup S_d$ with one pair of positive simple twist points, as shown in figure 3, we get a simple braided surface S of degree d over X, with $ed(d-1)$ positive twist points and $\chi(S) = d(\chi(X) - e(d-1))$. On the other hand, we can easily add to S pairs of opposite simple twist points, as shown in figure 4, in order to arbitrarily increase the number of twist points of S and decrease the Euler characteristic $\chi(S)$.

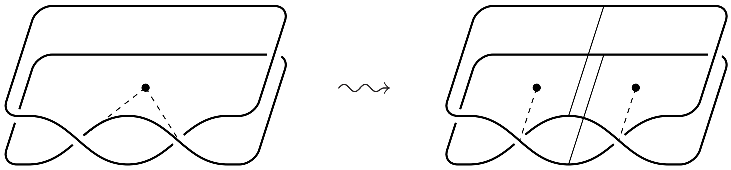

FIGURE 3.

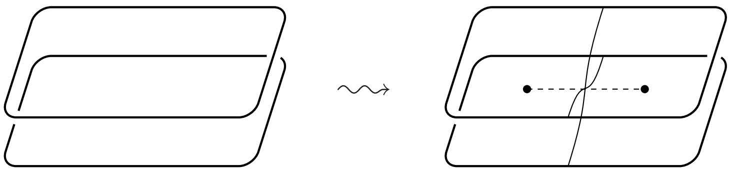

FIGURE 4.

We conclude this section by computing the Euler number $e(S)$ of the braided surface S, that is the self-intersection number of S in the oriented 4-manifold N, which coincides with the self-intersection of S as a multi-valued section of f.

PROPOSITION 2.3. *If N is oriented and $S \subset N$ is any braided surface over a closed surface X as above, then $e(S) = i(S) + ed = ed^2$.*

Proof. Let $s : X \to N$ be a cross section of f transverse to the null section. We can assume that S is simple and that the zeroes of s are not branch points. By translating $s(x)$ at every point in $S \cap f^{-1}(x)$ for every $x \in X$ and taking normal component with respect to S, we get a normal vector field v along S with non-degenerate singularities.

A point $y \in S$ is a singular point for v iff $f(y)$ is a singular point for s or y is a twist point for S; furthermore all the signs are coherent. Therefore we have $e(S) = i(S) + ed$. Then, the statement follows by proposition 2.1. $\square$

We notice that the results above can be easily generalized to the case of singular braided surfaces with transversal double points, by taking account of each double point as a pair of twist points. Namely, denoting by $n(S)$ the algebraic number of double points of S, we have $i(S) + 2n(S) = ed(d-1)$ and $e(S) = ed^2 - 2n(S)$.

3. Non-orientable braided surfaces in S^4

In this section we apply the results of the previous one, in order to show that the Viro-Kamada's representation theorem of orientable surfaces in S^4 as braided surfaces (cf. [8]) cannot be extended to include the non-orientable case.

In fact, by combining the results of the previous section with the Whitney's conjecture on non-orientable surfaces in S^4, proved by Massey in [14], we get very restrictive conditions for such a surface to be isotopic to a braided one, with respect to any reasonable definition of non-orientable braided surface. We recall that the Whitney conjecture imposes the following constrains to the self-intersection number e of a non-orientable surface of Euler characteristic χ in S^4: $e \equiv 2\chi \bmod 4$ and $|e| \leq 4 - 2\chi$.

It is natural to call a *non-orientable braided surface* in S^4 any non-orientable surface $S \subset S^4$ which is contained as a braided surface over X in the normal fiber bundle $\nu : N \to X$ of some fixed standard smooth non-orientable surface $X \subset S^4$, where N is identified with an open tubular neighborhood of X in S^4.

The most significant choice for X is the Veronese surface $V \subset S^4$ defined in the following way. First of all, we consider the space $\mathcal{M} \cong R^9$ of the 3×3 matrices over R with the inner product given by $\langle A, B\rangle = \mathrm{tr}(AB^T)$ for all $A, B \in \mathcal{M}$, and the map $\varphi : S^2 \to \mathcal{M}$ defined by $\varphi(x) = x^T x$ for any $x \in S^2 \subset R^3$. Since $\varphi(y) = \varphi(x)$ iff $y = \pm x$, we get an induced embedding $\psi : P^2 \to \mathcal{M}$, where P^2 is thought as the quotient of S^2 by the action of the antipodal map $x \mapsto -x$. Then, we put $V = \psi(P^2)$ after having identified S^4 with the intersection of the unit sphere of $\mathcal{M}$ with the affine subspace $\mathcal{L} = \{M \in \mathcal{M} \mid M = M^T \text{ and } \mathrm{tr}\, M = 1\} \cong R^5$.

The remarkable property that characterizes V is the existence of a symmetric splitting $S^4 \cong \overline{N} \cup_f \overline{N}$, where $\overline{N}$ is a closed tubular neighborhood of V in S^4 and f is an involution of $\mathrm{Bd}\, \overline{N}$ onto itself (see [13] and [16]). Such splitting has several relevant geometric properties (cf. [1] and [16]), moreover, from a topological point of view, it is related to the identification of S^4 with the quotient of the complex projective plane under complex conjugation, being V the branch set of the canonical projection $CP^2 \to CP^2/{\sim} \cong S^4$ (cf. [12], [13] and [15]).

Corollary 3.1. *Any non-orientable braided surface $S \subset S^4$ of degree d over the Veronese surface $V \subset S^4$ satisfies the following conditions: $\chi(S) \leq d(3 - 2d)$, $i(S) = 2d(d-1)$ and $e(S) = 2d^2$. As a consequence, the only surface S braided over V with $\chi(S) = 1$ is V itself (up to isotopy of braided surfaces) and there is no surface S braided over V with $\chi(S) = 0, -1, -3, -5, -7$.*

Proof. The first part of the corollary immediately follows from the results of the previous section, by taking into account that $e(V) = 2$. Now, the first inequality implies that: for $\chi(S) = 1$ we have $d = 1$, that is $S \cong V$; on the other hand, for $d \geq 2$ we have $\chi(S) \leq -2$; furthermore, we have $d \geq 3$, that gives us $\chi(S) \leq -9$, if we assume $\chi(S)$ odd and less than 1, since in this case also d is odd, because of the congruence $2d^2 \equiv 2\chi(S) \bmod 4$. $\square$

We remark that, by the equation $e(S) = 2d^2$, any surface $S \subset S^4$ braided over V has positive self-intesection. In order to get negative (resp. vanishing) self-intersection numbers, one could consider surfaces braided over $V' = \alpha(V)$ (resp. $V \# V'$), where $\alpha : S^4 \to S^4$ is the antipodal map.

Moreover, it is worth observing that, in the non-vanishing cases, only few values of the self-intersection among the ones allowed by the Whitney conjecture are realized by surfaces braided over V or V'. In fact, the self-intersection number of such a surface, besides having the very special form $e(S) = \pm 2d^2$, is bounded by the inequality $|e(S)| \leq 9/4 - \chi(S) + 3/4\,\sqrt{9 - 8\,\chi(S)}$, that can be derived from the inequality of corollary 3.1 by a straightforward computation.

However, the following problem remains still open: *is it possible to represent any orientable closed smooth 4-manifold as a cover of S^4 branched over a (possibly non-orientable) braided surface?*

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