ON THE GENERIC TRIANGLE GROUP
AND THE FREE METABELIAN GROUP OF RANK 2

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Abstract

We introduce the concept of a generic Euclidean triangle τ and study the group G_τ generated by the reflection across the edges of τ. In particular, we prove that the subgroup T_τ of all translations in G_τ is free abelian of infinite rank, while the index 2 subgroup H_τ of all orientation preserving transformations in G_τ is free metabelian of rank 2, with T_τ as the commutator subgroup. As a consequence, the group G_τ cannot be finitely presented and we provide explicit minimal infinite presentations of both H_τ and G_τ. This answers in the affirmative the problem of the existence of a minimal presentation for the free metabelian group of rank 2. Moreover, we discuss some examples of non-trivial relations in T_τ holding for given non-generic triangles τ.

1. INTRODUCTION

The term *triangle group* is generally reserved in the literature to the group G_τ generated by the reflections r_1, r_2, r_3 across the sides of an Euclidean, spherical or hyperbolic triangle τ with internal angles $\alpha_i = \pi/n_i$, where the specific geometry depends on $1/n_1 + 1/n_2 + 1/n_3$ being $= 1, > 1$ or < 1, respectively. The structure of these groups is well understood since the seminal works by Fricke and Klein [6] and Coxeter [4] (see also [5] and [9]). In particular, based on the fact that the triangle τ tiles the plane or the sphere, we have the finite presentation $G_\tau = \langle x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, (x_2 x_3)^{n_1}, (x_3 x_1)^{n_2}, (x_1 x_2)^{n_3} \rangle$ with the symbol x_i corresponding to the reflection r_i.

The hyperbolic case has been widely studied, with a special focus on its complex version, and several notions of generalized triangle groups have been considered, in terms of finite presentation independently on the original geometric setting.

But little seems to be known about the group G_τ for more general triangles τ, even if the strong hypothesis on the angles α_i is just mildly relaxed to include rational multiples of π, that is in the case of τ a *rational triangle* with $\alpha_i = m_i \pi / n_i$.

In this note we study the group G_τ for an arbitrary Euclidean triangle τ, starting from the case of *generic triangles*. These are introduced in Section 2 as the triangles whose edge lengths are algebraically independent over the rationals (up to a common factor), and can be considered as the opposite to the rational triangles in the spectrum of all Euclidean triangles. In particular, we show that generic triangles are included in the larger set of *typical triangles*, the ones whose angles are linearly independent over the rationals.

Our main results on the structure of the *triangle group* G_τ for a generic triangle τ, are presented in Sections 4 and 5, after a brief discussion of some generalities about G_τ and its linearization $S_\tau \subset O(2)$ in Section 3. They concern the *translation subgroup* T_τ, consisting of all translations in G_τ, and the *rotation subgroup* H_τ, the index 2 subgroup of

all orientation preserving transformations in G_τ. Namely, we show that T_τ is free abelian of infinite rank generated by the translation $t_1 = (r_1 r_2 r_3)^2$ and its conjugates in G_τ (Theorem 4.1), while H_τ is free metabelian of rank 2 generated by the rotations $r_2 r_1$ and $r_1 r_3$ (Theorem 5.1), with $[H_\tau, H_\tau] = T_\tau$. As a consequence, G_τ cannot be finitely presented. Moreover, we provide explicit presentations for H_τ (Theorem 5.2) and G_τ (Theorem 5.3), which are minimal in the sense that no relation can be removed without changing the group (Theorem 5.4). From the purely group theoretical viewpoint, this solves the problem of finding a minimal presentation for the free metabelian group of rank 2 (cf. [2]).

Finally, in Section 6 we discuss some examples of non-trivial relations in T_τ, holding for continuous families of non-generic but typical triangles τ and for certain isolated such triangles, respectively.

2. GENERIC TRIANGLES

Given an Euclidean triangle $\tau = A_1 A_2 A_3$, let $\ell_i > 0$ denote the length of the edge $e_i = A_j A_k$ and $\alpha_i > 0$ denote the measure in radians of the (non-oriented) interior angle $A_j A_i A_k$, with $\{i, j, k\} = \{1, 2, 3\}$.

Definition 2.1. We call τ a *generic triangle* if for some $k > 0$ (hence for almost every $k \in \mathbb{R}$) the real numbers $k\ell_1, k\ell_2$ and $k\ell_3$ are *algebraically independent* (over the rationals), namely it does not exist any non-trivial polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$ such that $p(k\ell_1, k\ell_2, k\ell_3) = 0$.

A different formulation of the above condition is that for every non-trivial polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$ the polynomial $q(x) = p(\ell_1 x, \ell_2 x, \ell_3 x)$ is non-trivial in $\mathbb{R}[x]$, or equivalently the field $\mathbb{Q}(\ell_1 x, \ell_2 x, \ell_3 x)$ has transcendence degree 3 over $\mathbb{Q}$ (see [10, Sec. 6.4]). Then, a straightforward argument on cardinalities explains why "some k" could be replaced by "almost every k" in the definition. Moreover, as a consequence of this reformulation of genericity, we have the following proposition.

Proposition 2.2. *The set of generic triangles is a dense G_δ-subset in the space of all Euclidean triangles.*

Proof. Due to the apparent invariance of genericity, we can argue on the space of triangles up to similitude, considered as a subspace of the real projective plane P^2, by identifying any Euclidean triangle τ as above with $[\ell_1 : \ell_2 : \ell_3] \in P^2$. In this context, for any given non-null polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$, the set of $(\ell_1, \ell_2, \ell_3) \in R^3$ such that $q(x) = p(\ell_1 x, \ell_2 x, \ell_3 x)$ is null in $\mathbb{R}[x]$, is characterized by a non-trivial system of homogeneous algebraic equations (expressing that all the coefficients of $q(x)$ vanish). Therefore, the corresponding subset of P^2 is a proper analytic subset, hence its complement is open and dense in P^2. Then, one can easily conclude by applying Baire's theorem. $\square$

The next proposition just translates Definition 2.1 in terms of trigonometric functions of the interior angles of the triangle τ.

Proposition 2.3. *An Euclidean triangle τ as above is a generic triangle if and only if one of the following equivalent properties holds:*

(s) *for some (hence almost every) $k \in \mathbb{R}$ the real numbers $k \sin \alpha_1, k \sin \alpha_2$ and $k \sin \alpha_3$ are algebraically independent over the rationals;*

(c) *for some (hence almost every) $k \in \mathbb{R}$ the real numbers $k\cos\alpha_1, k\cos\alpha_2$ and $k\cos\alpha_3$ are algebraically independent over the rationals.*

Proof. The equivalence of the condition in Definition 2.1 with property (s) immediately follows by the law of sines, while some work is needed to verify the equivalence between (s) and (c). Once these properties are reformulated in terms of extensions of $\mathbb{Q}$ involving the indeterminate x as above, we are reduced to proving that $\mathbb{Q}(x\sin\alpha_1, x\sin\alpha_2, x\sin\alpha_3)$ and $\mathbb{Q}(x\cos\alpha_1, x\cos\alpha_2, x\cos\alpha_3)$ have the same transcendence degree over $\mathbb{Q}$. By elementary trigonometry, from $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ we get the equations

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 + 2\cos\alpha_1\cos\alpha_2\cos\alpha_3 = 1\,, \tag{1}$$

$$\begin{aligned}\sin^4\alpha_1 + \sin^4\alpha_2 + \sin^4\alpha_3 + 4\sin^2\alpha_1\sin^2\alpha_2\sin^2\alpha_3 + \\ - 2\sin^2\alpha_1\sin^2\alpha_2 - 2\sin^2\alpha_1\sin^2\alpha_3 - 2\sin^2\alpha_2\sin^2\alpha_3 = 0\,.\end{aligned} \tag{2}$$

Multiplying (1) by x^3, we see that x is algebraic over $\mathbb{Q}(x\cos\alpha_1, x\cos\alpha_2, x\cos\alpha_3)$, hence the possibly larger extension $\mathbb{Q}(x\cos\alpha_1, x\cos\alpha_2, x\cos\alpha_3, x)$ has the same transcendence degree as $\mathbb{Q}(x\cos\alpha_1, x\cos\alpha_2, x\cos\alpha_3)$ over $\mathbb{Q}$. Similarly, multiplying (2) by x^6, we see that x is algebraic over $\mathbb{Q}(x\sin\alpha_1, x\sin\alpha_2, x\sin\alpha_3)$ as well, hence the transcendence degree of $\mathbb{Q}(x\sin\alpha_1, x\sin\alpha_2, x\sin\alpha_3, x)$ over $\mathbb{Q}$ is the same as that of $\mathbb{Q}(x\sin\alpha_1, x\sin\alpha_2, x\sin\alpha_3)$. Now, the relations $x^2\sin^2\alpha_i + x^2\cos^2\alpha_i = x^2$ allow us to conclude that $\mathbb{Q}(x\cos\alpha_1, x\cos\alpha_2, x\cos\alpha_3, x)$ and $\mathbb{Q}(x\sin\alpha_1, x\sin\alpha_2, x\sin\alpha_3, x)$ have the same transcendence degree over $\mathbb{Q}$. $\square$

As noticed in the above proof, due to the relation $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ we cannot have $k = 1$ (or any rational number) in points (s) and (c) of Proposition 2.3. However, if τ is a generic triangle then ℓ_1, ℓ_2, ℓ_3, as well as $\sin\alpha_1, \sin\alpha_2, \sin\alpha_3$ and $\cos\alpha_1, \cos\alpha_2, \cos\alpha_3$, form linearly independent triples over the rationals, since linear independence is a homogeneous condition where the factor $r > 0$ can be canceled.

Lemma 2.4. *If an algebraic relation $p(\sin\alpha_1, \sin\alpha_2, \sin\alpha_3, \cos\alpha_1, \cos\alpha_2, \cos\alpha_3) = 0$, with $p(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}[x_1, x_2, x_3, x_4, x_5, x_6]$, holds for a generic triangle, then it holds for every triangle.*

Proof. This is a direct consequence of Proposition 2.2. $\square$

A deeper analysis of the algebraic dependence of $\cos\alpha_1, \cos\alpha_2$ and $\cos\alpha_3$, shows that generic triangles are typical, in the sense of the following definition (see [7, Sec. 6.B]).

Definition 2.5. *An Euclidean triangle τ as above is called a typical triangle if the real numbers α_1, α_2 and α_3 are linearly independent over the rationals.*

Proposition 2.6. *Generic Euclidean triangles are typical.*

Proof. Let τ a generic triangle. We want to prove that equation (1) is essentially the only algebraic relation between the cosines of the interior angles of τ, that is any polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$ with $p(\cos\alpha_1, \cos\alpha_2, \cos\alpha_3) = 0$ is divisible by

$$x_1^2 + x_2^2 + x_3^2 + 2x_1x_2x_3 - 1\,. \tag{3}$$

Assume that the identity $p(\cos\alpha_1, \cos\alpha_2, \cos\alpha_3) = 0$ holds for τ. Then, according to Lemma 2.4, it must hold for any triangle, and by using once again the relation

$\alpha_1 + \alpha_2 + \alpha_3 = \pi$ we have $p(-\cos(\alpha_2 + \alpha_3), \cos\alpha_2, \cos\alpha_3) = 0$ for every $\alpha_2, \alpha_3 \geq 0$ such that $\alpha_2 + \alpha_3 < \pi$. Therefore, $p(x_1, x_2, x_3)$ is divisible by both the linear binomials $x_1 + x_2 x_3 \pm \sqrt{(1 - x_2^2)(1 - x_3^2)}$ in the indeterminate x_1 with coefficients in the quadratic closure of the field of fractions $\mathbb{Q}(x_2, x_3)$, hence it is divisible by $x_1^2 + x_2^2 + x_3^2 + 2x_1 x_2 x_3 - 1$ in $\mathbb{Z}[x_1, x_2, x_3]$ (notice that the last polynomial is monic with respect to x_1).

Now, by contradiction, let $n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = 0$ be a non-trivial vanishing linear combination of the interior angles α_1, α_2 and α_3 of a generic triangle, with integral coefficients n_1, n_2 and n_3, which can be assumed coprime without loss of generality. By elementary trigonometry, we have

$$\cos(n_1\alpha_1) - \cos(n_2\alpha_2)\cos(n_3\alpha_3) = \sin(n_2\alpha_2)\sin(n_3\alpha_3).$$

Squaring and using the Pythagorean identity, we readily obtain

$$T_{n_1}^2(\cos\alpha_1) + T_{n_2}^2(\cos\alpha_2) + T_{n_3}^2(\cos\alpha_3) - 2\,T_{n_1}(\cos\alpha_1)T_{n_2}(\cos\alpha_2)T_{n_3}(\cos\alpha_3) = 1\,,$$

where $T_n(x) \in \mathbb{Z}[x]$ denotes the n-th Chebyshev polynomial defined by the identity $T_n(\cos\alpha) = \cos(n\alpha)$. By the above, the polynomial

$$T_{n_1}^2(x_1) + T_{n_2}^2(x_2) + T_{n_3}^2(x_3) - 2\,T_{n_1}(x_1)T_{n_2}(x_2)T_{n_3}(x_3) - 1 \tag{4}$$

must be divisible by (3). Since $T_n(1) = 1$ for all n, setting $x_2 = x_3 = 1$ both in (3) and (4), we get $(x_1 + 1)^2 = 0$ and $(T_{n_1}(x_1) - 1)^2 = 0$, respectively. Thus, $T_{n_1}(x_1) - 1$ must be divisible by $x_1 + 1$ in $\mathbb{Z}[x_1]$. This implies that n_1 is even, because $T_n(x)$ has same parity of n. By the symmetry of (3) and (4), the same argument shows that n_2 and n_3 must be even as well, contradicting the coprimality assumption. $\square$

In the light of Proposition 2.6, generic triangles can be somewhat thought of as the opposite end in the spectrum of all Euclidean triangles with respect to the rational ones.

3. The triangle group

For any Euclidean triangle τ we denote by $G_\tau = \langle r_1, r_2, r_3 \rangle \subset \mathrm{E}(2)$ the subgroup of the group $\mathrm{E}(2)$ of the Euclidean isometries of the plane generated by the reflections r_1, r_2 and r_3 across the edges e_1, e_2 and e_3 of τ, respectively. We call G_τ the *triangle group* of τ.

The standard exact sequence

$$1 \longrightarrow \mathbb{R}^2 \xrightarrow{\iota} \mathrm{E}(2) \xrightarrow{\lambda} \mathrm{O}(2) \longrightarrow 1\,,$$

where ι is the inclusion of $\mathbb{R}^2$ in $\mathrm{E}(2)$ as the subgroup of translations and λ is the linearization homomorphism, induces by restriction the exact sequence

$$1 \longrightarrow T_\tau \xrightarrow{\iota_\tau} G_\tau \xrightarrow{\lambda_\tau} S_\tau \longrightarrow 1\,, \tag{5}$$

where $T_\tau \subset G_\tau$ is the *translation subgroup* consisting of all translations in G_τ, while $S_\tau = \lambda(G_\tau) = \langle s_1, s_2, s_3 \rangle \subset \mathrm{O}(2)$ with $s_i = \lambda(r_i)$ the linearization of r_i.

We observe that the structure of the group G_τ (including the latter exact sequence) is invariant under similarities. Hence, without loss of generality we can assume that the

incircle of the triangle τ is coincides with the unit circle centered at the origin. Under this assumption, we have

$$(x)r_i = (x)s_i + 2v_i \tag{6}$$

for every $x \in \mathbb{R}^2$ (here and in the following we use the right notation for the action of G_τ), where v_i is the unit vector from the origin to the tangency point of the edge e_i and the incircle of τ, as shown in Figure 1.

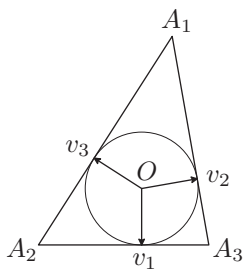Figure 1. The unit vectors v_1, v_2 and v_3.

We want to determine a minimal presentation of the group S_τ in the case when τ is a typical, and hence S_τ is a dense subgroup of O(2). In order to do that, we first recall from [7, Sec. 6.A] the definition of a stable sequence and the stability criterion for a product of generators of S_τ to be trivial.

Definition 3.1. A sequence $i_1 i_2 \ldots i_n$ of symbols from $\{1, 2, \ldots, N\}$ is called a *stable sequence* if its terms can be paired into disjoint pairs of identical symbols, one located at an odd and the other at an even position. Differently said, the length n of the sequence is even and the symbolic alternating sum $i_1 - i_2 + \ldots + i_{n-1} - i_n$ vanish (as an algebraic sum of symbols, not of integers).

The motivation for the term "stable" is that a stable sequence as above represents the sequence of sides of a polygonal billiard (whose N sides are arbitrarily numbered) visited by a n-periodic trajectory, which is stable in the sense that it survives to small perturbations of the polygon (see [7, Sec. 6.A]). Actually, also Lemma 3.3 below and its proof are essentially translated from the context of stable trajectories in polygonal billiards, focusing on the case $N = 3$.

Before going on, let us give an operational characterization of stability.

Lemma 3.2. *A sequence $i_1 i_2 \ldots i_n$ is stable if and only if one can reduce it to the empty sequence by a finite number of operation of the following types:*

(a) *transposition of two adjacent subsequences both consisting of two symbols;*

(b) *deletion of a subsequence consisting of two identical symbols.*

Proof. First of all, we note that both operations and the inverse of the second one, that is the insertion of two adjacent identical symbols in a sequence, all preserve the parity of the position of each term in the sequence, hence they preserve stability. This immediately gives the "if" part of the statement, since the empty sequence is stable.

The "only if" part can be proved by induction on the length of the sequence, starting once again from the empty sequence. For the inductive step, assume we are given any non-empty stable sequence $i_1 i_2 \ldots i_n$. The stability implies that $i_{2k-1} = i_2$ for some

$1 \leq k \leq n/2$. If $k = 1$, we can reduce the length of the sequence by deleting the subsequence $i_1 i_2$. Otherwise, by $k - 2$ transpositions of pairs, we get a sequence starting with the four symbols $i_1 i_2 i_{2k-1} i_{2k}$, then we can reduce the length of the word by deleting the subsequence $i_2 i_{2k-1}$. $\square$

Lemma 3.3. *If a sequence $i_1 i_2 \ldots i_n$ of symbols from $\{1, 2, 3\}$ is stable, then the product $s_{i_1} s_{i_2} \ldots s_{i_n}$ is the identity in S_τ. Moreover, for a typical triangle τ the stability of the sequence $i_1 i_2 \ldots i_n$ is also necessary in order $s_{i_1} s_{i_2} \ldots s_{i_n}$ to be the identity.*

Proof. We proceed in the same spirit as in [7, Sec. 6.B]. We first orient the egdes e_1, e_2 and e_3 in the counterclockwise way along the boundary of the triangle τ, and denote by β_i the oriented angle from e_1 (fixed as reference vector) to e_i. Then, we have $\beta_1 = 0$, $\beta_2 = \pi - \alpha_3$ and $\beta_3 = \pi + \alpha_2 \bmod 2\pi$. Moreover, any composition $s_j s_k$ gives the linear rotation of angle $2(\beta_k - \beta_j) \bmod 2\pi$, and hence any product $s_{i_1} s_{i_2} \ldots s_{i_n}$ with n even gives the linear rotation of angle

$$\varphi = 2(\beta_{i_2} - \beta_{i_1}) + \cdots + 2(\beta_{i_n} - \beta_{i_{n-1}}) \bmod 2\pi. \tag{7}$$

Now, the stability of the sequence $i_1 i_2 \ldots i_n$ implies that $\varphi = 0 \bmod 2\pi$ and thus $s_{i_1} s_{i_2} \ldots s_{i_n}$ is the identity in S_τ.

In the opposite direction, start with a product $s_{i_1} s_{i_2} \ldots s_{i_n}$ that gives the identity. Then, n must be even and (7) can be rewritten in terms of the α_i's by using the above identities. If the sequence $i_1 i_2 \ldots i_n$ is not stable, this yields a non-trivial rational linear relation among the angles α_2, α_3 and π, which implies that τ is not typical. $\square$

At this point, we are in position to obtain the wanted presentation of S_τ.

Proposition 3.4. *For a typical triangle τ the group S_τ admits the finite presentation*

$$\langle x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, (x_1 x_2 x_3)^2 \rangle,$$

with the symbols x_1, x_2, x_3 corresponding to s_1, s_2, s_3, respectively.

Proof. According to Lemma 3.3, all the four relations of the presentation hold in S_τ, because the corresponding sequences of indices are stable.

Viceversa, Lemmas 3.2 and 3.3 say that any word $x_{i_1} x_{i_2} \ldots x_{i_n}$ representing the identity in S_τ can be reduced to the empty word by canceling squared terms x_i^2 and commuting products $x_i x_j$ and $x_k x_l$. So, to conclude the proof it is enough to show that any commutator $[x_i x_j, x_k x_l] = (x_i x_j)^{-1} (x_k x_l)^{-1} (x_i x_j)(x_k x_l)$ is the identity modulo the given four relations. Up to inversions, the only non-trivial cases are $[x_1 x_2, x_1 x_3], [x_1 x_2, x_2 x_3]$ and $[x_1 x_3, x_2 x_3]$. For these we have: $[x_1 x_2, x_1 x_3] = x_3 (x_1 x_2 x_3)^{-2} x_3$, $[x_1 x_2, x_2 x_3] = x_2 x_1 (x_1 x_2 x_3)^{-2} x_1 x_2$ and $[x_1 x_3, x_2 x_3] = x_3 x_1 (x_1 x_2 x_3)^{-2} x_1 x_3$. $\square$

4. The translation subgroup

Due to Lemma 3.3 and the exact sequence (5), for any Euclidean triangle τ the product $r_{i_1} r_{i_2} \ldots r_{i_n}$ gives a translation in T_τ if the sequence $i_1 i_2 \ldots i_n$ is stable. On the other hand, when the triangle τ is typical we obtain in this way all the translations in T_τ, and it is clear from Proposition 3.4 that a special role is played by the minimal stable product $t_1 = (r_1 r_2 r_3)^2$, coming from the code-word of the Fagnano trajectory, the simplest stable

periodic trajectory in any acute triangle (see [7, Sec. 2.A]). Notice that the translation t_1 is non-trivial for any (non-degenerate) triangle τ, as it easily follows by elementary geometry.

We denote the conjugation class of t_1 in G_τ by

$$C(t_1) = \{((t_1))g = g^{-1}t_1 g \mid g \in G_\tau\} \subset T_\tau.$$

Notational warning. *We are aware that the adopted notation for the conjugates is not the standard one, but we believe that it makes more readable the equations displayed in the sequel, where the conjugating element is in most cases a long product. For simplicity sake, we avoid the use of brackets to enclose such products. To resolve the possible ambiguity in the interpretation of the expression $((a))bc$, we specify that this will always mean the conjugate of a by bc, that is $(bc)^{-1}a(bc)$, and not instead the product $(b^{-1}ab)c$, which will be written as $((a))b \cdot c$.*

According to equation (6), by identifying translations in T_τ with the corresponding vectors in $\mathbb{R}^2$ and considering the natural action of $S_\tau \subset O(2)$ on them, for $g = r_{i_1} r_{i_2} \ldots r_{i_n} \in G_\tau$ the conjugate $((t_1))g$ is given by

$$((t_1))r_{i_1} r_{i_2} \ldots r_{i_n} = ((t_1))s_{i_1} s_{i_2} \ldots s_{i_n}. \tag{8}$$

In the case when τ is a typical triangle, the density of the subgroup $S_\tau \subset O(2)$ implies that $C(t_1)$ forms a dense subset of the circle $\rho S^1 \subset R^2$ of radius

$$\rho = \|t_1\| = 4(\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3).$$

The translation t_1 and its conjugates $t_2 = (r_2 r_3 r_1)^2 = ((t_1))r_1$ and $t_3 = (r_3 r_1 r_2)^2 = ((t_1))r_3$, are represented in Figure 2 (here we assume the same numbering as in the previous Figure 1 for the edges of the triangle τ).
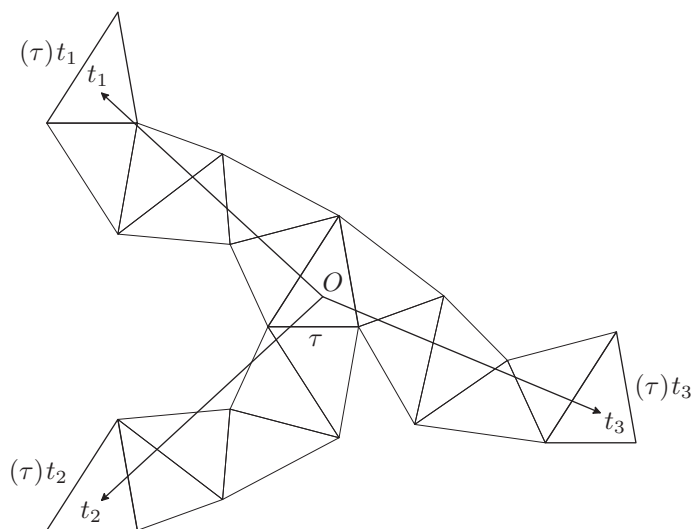

Figure 2. The translations t_1, t_2 and t_3

Theorem 4.1. *For a typical triangle τ the translation subgroup $T_\tau \subset G_\tau$ is normally generated by the translation $t_1 = (r_1 r_2 r_3)^2$. Moreover, if τ is generic then T_τ is a free abelian group having as a basis the conjugation class $C(t_1)$.*

Proof. Given any $t \in T_\tau$ with τ a typical triangle, we can express it as a product $r_{i_1} r_{i_2} \ldots r_{i_n}$ of generators of G_τ. In view of the exact sequence (5), the corresponding product $s_{i_1} s_{i_2} \ldots s_{i_n}$ gives the identity in S_τ, hence the sequence $i_1 i_2 \ldots i_n$ is stable by Lemma 3.3. Then, arguing as in the proof of Proposition 3.4, we can rewrite $r_{i_1} r_{i_2} \ldots r_{i_n}$ as a product of conjugates of r_1^2, r_2^2, r_3^2 and $(r_1 r_2 r_3)^2$. Since the r_i^2's are trivial in G_τ, we can conclude that t is a product of conjugates of t_1, which gives the first part of the theorem.

Now, assume that τ is a generic triangle. We have to show that $C(t_1)$ is linearly independent over $\mathbb{Z}$, that is the only vanishing linear combination of pairwise distinct elements of $C(t_1)$ with integral coefficients is the trivial one.

Consider any vanishing linear combination

$$\sum_{j=1}^m k_j \, (\!(t_1)\!) r_{i_{j,1}} r_{i_{j,2}} \ldots r_{i_{j,n_j}} = 0 \tag{9}$$

of pairwise distinct conjugates $(\!(t_1)\!) r_{i_{j,1}} r_{i_{j,2}} \ldots r_{i_{j,n_j}}$ of t_1, with coefficients $k_1, \ldots, k_m \in \mathbb{Z}$. Since $(\!(t_1)\!) r_1 r_2 r_3 = t_1$, without loss of generality we assume that all the n_j are even. Then, equation (7) tells us that the oriented angle from t_1 to $(\!(t_1)\!) r_{i_{j,1}} r_{i_{j,2}} \ldots r_{i_{j,n_j}}$ equals

$$2(\beta_{i_{j,2}} - \beta_{i_{j,1}}) + \ldots + 2(\beta_{i_{j,n_j}} - \beta_{i_{j,n_j-1}}) \bmod 2\pi \,, \tag{10}$$

where β_i denotes the oriented angle from e_1 to e_i, namely $\beta_1 = 0$, $\beta_2 = \pi - \alpha_3$ and $\beta_3 = \pi + \alpha_2 \bmod 2\pi$. This can also be written in the form

$$m_{j,2}\alpha_2 + m_{j,3}\alpha_3 \bmod 2\pi \,, \tag{11}$$

with $m_{j,2}$ and $m_{j,3}$ even integers, hence the scalar product of (9) with t_1 gives

$$\sum_{j=1}^m k_j \cos(m_{j,2}\alpha_2 + m_{j,3}\alpha_3) = 0 \,, \tag{12}$$

where ρ^2, the squared norm of t_1 (and all its conjugates), has been collected as a common factor and canceled. Similarly, the scalar product of (9) with the vector obtained by rotating t_1 of $\pi/2$ radians gives

$$\sum_{j=1}^m k_j \sin(m_{j,2}\alpha_2 + m_{j,3}\alpha_3) = 0 \,. \tag{13}$$

Notice that in the above equations the pairs $(m_{j,2}, m_{j,3})$ are different from each other, in that the conjugates in (9) are pairwise distinct. Moreover, possibly after suitable changes of signs in order to have either $m_{j,2} > 0$ or $m_{j,2} = 0$ and $m_{j,3} \geq 0$, we can collect the (at most two) terms corresponding to opposite pairs.

According to Lemma 2.4, the identities (12) and (13) hold for every triangle, that is for every $\alpha_2, \alpha_3 > 0$ such that $\alpha_2 + \alpha_3 < \pi$. Therefore, the linear independence over the reals of the complex functions $(x_1, x_2) \mapsto \exp(i(m_1 x_1 + m_2 x_2))$ with $m_1 > 0$ or $m_1 = 0$ and $m_2 \geq 0$, allows us to conclude that $k_j = 0$ for every $j = 1, \ldots, m$. $\square$

In view of the above proof, if τ is a generic triangle then any conjugate $t \in C(t_1)$ can be obtained from t_1 by a rotation of $2m\alpha_2 - 2n\alpha_3$ radians for some (uniquely determined,

as τ is typical) integers m and n, hence $t = ((t_1))(r_1 r_2)^n (r_1 r_3)^m$ according to equations (7) and (8). Therefore, for a generic triangle τ we can write

$$C(t_1) = \{t_{n,m} = ((t_1))(r_1 r_2)^n (r_1 r_3)^m, \ n, m \in \mathbb{Z}\}. \tag{14}$$

Based on Theorem 4.1, Theorems 4.2 and 5.1 provide an infinite presentation of G_τ for a generic triangle τ and show that no such a finite presentation can exist. A minimal presentation of G_τ will be given in Section 5.

Theorem 4.2. *For a generic triangle τ the group G_τ admits the presentation*

$$\langle x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, [w, ((w))(x_1 x_2)^n (x_1 x_3)^m], \ n, m \in \mathbb{Z}, (n,m) \neq (0,0)\rangle,$$

with $w = (x_1 x_2 x_3)^2$ and the symbols x_1, x_2, x_3 corresponding to r_1, r_2, r_3, respectively.

Proof. By a standard argument (see [8, Sec. 10.2]), a presentation of G_τ can be derived from presentations of the groups T_τ and S_τ involved in the exact sequence (5). In view of Proposition 2.6 a presentation of S_τ is given by Proposition 3.4, while T_τ is free abelian on the set of generators (14), according to Theorem 4.1.

Pulling back the generators s_i of S_τ to the generators r_i of G_τ, the relations $s_i^2 = 1$ still hold in the same form $r_i^2 = 1$, while the relation $(s_1 s_2 s_3)^2 = 1$ turns into the identity $(r_1 r_2 r_3)^2 = t_1 = t_{0,0}$. Moreover, based on (14), for the generators of T_τ we have

$$t_{n,m} = ((t_{0,0}))(r_1 r_2)^n (r_1 r_3)^m = (((r_1 r_2 r_3)^2))(r_1 r_2)^n (r_1 r_3)^m. \tag{15}$$

At this point, to complete the set of relations for G_τ it remains to rewrite in the generators r_i's, by using equation (15), the commutators $[t_{n,m}, t_{n',m'}]$ and the equations

$$((t_{n,m}))r_1 = t_{-n+1,-m-1}, \quad ((t_{n,m}))r_2 = t_{-n+2,-m-1}, \quad ((t_{n,m}))r_3 = t_{-n+1,-m}, \tag{16}$$

which express in terms of the $t_{n,m}$'s their conjugates by the r_i's. The latter equations could be easily shown to hold, by taking into account equation (8) and the commutativity of SO(2), and by using the relations r_i^2 and the trivial identity $(((r_1 r_2 r_3)^2))r_1 r_2 r_3 = (r_1 r_2 r_3)^2$. However, we are going to validate them in a different way.

In fact, the rest of the proof is aimed to see how the rewriting of equations (16) in the r_i's, as well the rewriting of the commutators $[t_{n,m}, t_{n',m'}]$ with $(n', m') \neq (n, m)$, can be derived from the relations r_i^2 and the relations

$$[(r_1 r_2 r_3)^2, (((r_1 r_2 r_3)^2))(r_1 r_2)^n (r_1 r_3)^m] \tag{17}$$

with $n, m \in \mathbb{Z}$ and $(n, m) \neq (0,0)$, which represent the special commutators $[t_{0,0}, t_{n,m}]$.

We start by observing that the relations r_i^2 imply

$$[r_1 r_3, r_1 r_2] = r_1 r_3 r_1 r_2 r_3 r_1 r_2 r_1 = (((r_1 r_2 r_3)^2))(r_1 r_3)^{-1}, \tag{18}$$

which in turn, together with the relation $[(r_1 r_2 r_3)^2, (((r_1 r_2 r_3)^2))(r_1 r_2)^n (r_1 r_3)^m]$ conjugated by $(r_1 r_3)^{-1}$, implies

$$(((r_1 r_2 r_3)^2))(r_1 r_2)^n (r_1 r_3)^m r_1 r_2 = (((r_1 r_2 r_3)^2))(r_1 r_2)^n (r_1 r_3)^{m\mp 1} r_1 r_2 (r_1 r_3)^{\pm 1}.$$

Hence, by increasing/decreasing induction on m, based on the trivial case of $m = 0$,

$$(((r_1 r_2 r_3)^2))(r_1 r_2)^n (r_1 r_3)^m (r_1 r_2)^{\pm 1} = (((r_1 r_2 r_3)^2))(r_1 r_2)^{n\pm 1} (r_1 r_3)^m$$

for every $m \in \mathbb{Z}$. Finally, by increasing/decreasing induction on n, based on the trivial case of $n = 0$, we get

$$(((r_1r_2r_3)^2))(r_1r_3)^m(r_1r_2)^n = (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m \tag{19}$$

for every $n, m \in \mathbb{Z}$.

As a consequence of (19), the rewriting of any commutator $[t_{n,m}, t_{n',m'}]$ is equivalent up to conjugation to that of $[t_{0,0}, t_{n'-n,m'-m}]$. In fact,

$$[(((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m, (((r_1r_2r_3)^2))(r_1r_2)^{n'}(r_1r_3)^{m'}]$$

once conjugated by $(r_1r_3)^{-m}(r_1r_2)^{-n}$ becomes

$$[(r_1r_2r_3)^2, (((r_1r_2r_3)^2))(r_1r_2)^{n'}(r_1r_3)^{m'-m}(r_1r_2)^{-n}],$$

and this is equivalent to

$$[(r_1r_2r_3)^2, (((r_1r_2r_3)^2))(r_1r_2)^{n'-n}(r_1r_3)^{m'-m}]$$

by equation (19).

Moreover, we obtain the rewriting of the relations (16) from the relations r_i^2 and the relations (17), by the following chains of equalities, whose last step is based on two applications of equation (19):

$$\begin{aligned} (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m r_1 &= (((r_1r_2r_3)^2))r_1(r_1r_2)^{-n}(r_1r_3)^{-m} \\ &= (((r_1r_2r_3)^2))r_1r_2r_3r_1(r_1r_2)^{-n}(r_1r_3)^{-m} \\ &= (((r_1r_2r_3)^2))(r_1r_2)^{-n+1}(r_1r_3)^{-m-1}; \end{aligned}$$

$$\begin{aligned} (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m r_2 &= (((r_1r_2r_3)^2))r_1(r_1r_2)^{-n}(r_1r_3)^{-m}r_1r_2 \\ &= (((r_1r_2r_3)^2))r_1r_2r_3r_1(r_1r_2)^{-n}(r_1r_3)^{-m}r_1r_2 \\ &= (((r_1r_2r_3)^2))(r_1r_2)^{-n+2}(r_1r_3)^{-m-1}; \end{aligned}$$

$$\begin{aligned} (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m r_3 &= (((r_1r_2r_3)^2))r_1(r_1r_2)^{-n}(r_1r_3)^{-m}r_1r_3 \\ &= (((r_1r_2r_3)^2))r_1r_2r_3r_1(r_1r_2)^{-n}(r_1r_3)^{-m}r_1r_3 \\ &= (((r_1r_2r_3)^2))(r_1r_2)^{-n+1}(r_1r_3)^{-m}. \end{aligned}$$

This concludes the proof. $\square$

5. The rotation subgroup

By intersecting the exact sequence (5) with the group $\mathrm{SE}(2)$ of orientation preserving Euclidean isometries, we get the new exact sequence

$$1 \longrightarrow T_\tau \xrightarrow{\iota_\tau|} H_\tau \xrightarrow{\lambda_\tau|} R_\tau \longrightarrow 1, \tag{20}$$

where $H_\tau = G_\tau \cap \mathrm{SE}(2)$ is a subgroup of index 2 in G_τ, which we call the *rotation subgroup*, while $R_\tau = S_\tau \cap \mathrm{SE}(2) \subset \mathrm{SO}(2)$ is the abelian subgroup of linear rotations in S_τ. As an extension of an abelian group by another abelian group, H_τ is metabelian, hence G_τ is virtually metabelian.

If τ is a typical triangle, then R_τ is a dense free abelian subgroup of $SO(2)$ generated by the two rotations $s_1 s_2$ and $s_1 s_3$ (cf. proof of Lemma 3.3). Moreover, in the light of the first part of Theorem 4.1, equations (14) and (18) imply that T_τ coincides with the commutator subgroup $[H_\tau, H_\tau]$ when τ is typical.

Theorem 5.1. *For a generic triangle τ the rotation subgroup $H_\tau \subset G_\tau$ is a free metabelian group of rank 2 generated by the rotations $r_1 r_2$ and $r_1 r_3$, meaning that it is isomorphic to the metabelianization $F_2/[[F_2, F_2], [F_2, F_2]]$ of the free group F_2 on two generators corresponding to those rotations. As a consequence, G_τ does not admit any finite presentation.*

Proof. Since $F_2/[[F_2, F_2], [F_2, F_2]]$ is known not to admit any finite presentation [11] (cf. also [1]) and H_τ is a finite index subgroup of G_τ, the second part of the statement follows (see [8, Sec. 9.1]) once we know that $H_\tau \cong F_2/[[F_2, F_2], [F_2, F_2]]$.

The above observation that $[H_\tau, H_\tau] = T_\tau$ together with the second part of Theorem 4.1 and the fact that R_τ is free abelian of rank 2, would suffice to prove that H_τ is free metabelian of rank 2. However, for future reference, we give an explicit isomorphism $H_\tau \cong F_2/[[F_2, F_2], [F_2, F_2]]$ by way of a presentation of H_τ.

Starting from the presentation of G_τ given in Theorem 4.2 and applying the Reidemeister-Schreier method (see [8, Sec. 9.1]) to the subgroup H_τ with Schreier transversal $\{1, x_1\}$, we get the following presentation for H_τ

$$\langle y_2, y_3, z_1, z_2, z_3 \mid z_1, y_2 z_2, y_3 z_3, a_{n,m}, z_2 y_2, z_3 y_3, a'_{n,m}, \; n, m \in \mathbb{Z}, (n,m) \neq (0,0) \rangle,$$

where the generators are given by $y_2 = x_2 x_1^{-1}$, $y_3 = x_3 x_1^{-1}$, $z_1 = x_1^2$, $z_2 = x_1 x_2$, $z_3 = x_1 x_3$, and the relations $a_{n,m}$ and $a'_{n,m}$ are the transcriptions in terms of such generators of the commutator $[w, (\!(w)\!)(x_1 x_2)^n (x_1 x_3)^m]$ and its conjugate $(\!([w, (\!(w)\!)(x_1 x_2)^n (x_1 x_3)^m])\!)x_1^{-1} = [(\!(w)\!)x_1 z_1^{-1}, (\!(w)\!)x_1 (x_2 x_1)^n (x_3 x_1)^m z_1^{-1}]$, respectively, that is

$$a_{n,m} = [z_2 y_3 z_1 y_2 z_3, (\!(z_2 y_3 z_1 y_2 z_3)\!)z_2^n z_3^m],$$

$$a'_{n,m} = [(\!(y_2 z_3 z_2 y_3 z_1)\!)z_1^{-1}, (\!(y_2 z_3 z_2 y_3 z_1)\!)(y_2 z_1)^n (y_3 z_1)^m z_1^{-1}].$$

Now, we eliminate the generators z_1, z_2 and z_3, by using the relations $z_1, y_2 z_2$ and $y_3 z_3$, and then replace y_3 by y_3^{-1} to obtain the new presentation for H_τ

$$\langle y_2, y_3 \mid [[y_2, y_3], (\!([y_2, y_3])\!)y_2^n y_3^m], [[y_2^{-1}, y_3^{-1}], (\!([y_2^{-1}, y_3^{-1}])\!)y_2^n y_3^m], \; n, m \in \mathbb{Z}, (n,m) \neq (0,0) \rangle.$$

Finally, since H_τ is metabelian and all the relations in the above presentation of H_τ belong to $[[F_2, F_2], [F_2, F_2]]$, we can conclude that $H_\tau \cong F_2/[[F_2, F_2], [F_2, F_2]]$ with the rotations $r_1 r_2$ and $r_1 r_3$ corresponding to the free generators y_2^{-1} and y_3 of F_2, respectively. $\square$

In the following, we provide presentations of the groups H_τ and G_τ, which are minimal in the sense that no relation can be removed without changing the group. According to Theorem 5.1, this solves in the affirmative the problem of the existence of a minimal presentation of the free metabelian group $F_2/[[F_2, F_2], [F_2, F_2]]$ of rank 2 (cf. [2]).

The two presentations are given in Theorems 5.2 and 5.3, respectively, while their minimality is proved in Theorem 5.4. For H_τ we just refine the presentation considered in the proof of Theorem 5.1 to make it minimal. On the contrary, the minimal presentation of G_τ is derived from that of H_τ, and it has different relations with respect to the one given in Theorem 4.2. We denote by $<_{\mathrm{lex}}$ the lexicographic order.

Theorem 5.2. *For a generic triangle τ the group H_τ admits the presentation*

$$\langle y_2, y_3 \mid [[y_2, y_3], (\!([y_2, y_3])\!)y_2^n y_3^m], \, n, m \in \mathbb{Z}, (n, m) >_{\text{lex}} (0, 0)\rangle,$$

with the symbols y_2, y_3 corresponding to the rotations $r_2 r_1$ and $r_1 r_3$, respectively.

Proof. The relations in the statement imply that

$$(\!([y_2, y_3])\!)y_2^n y_3^m y_2 = (\!([y_2, y_3])\!)y_2^n y_3^{m\mp 1} y_2 y_3^{\pm 1}$$

for every $(n, m) >_{\text{lex}} (0, 0)$. From this family of equalities, arguing as in the proof of Theorem 5.1 when obtaining equation (19), by increasing/decreasing induction on m, based on the trivial case of $m = 0$, and then by induction on $n \geq 0$, based on the trivial case of $n = 0$, we get

$$(\!([y_2, y_3])\!)y_3^m y_2^n = (\!([y_2, y_3])\!)y_2^n y_3^m \tag{21}$$

for every $(n, m) >_{\text{lex}} (0, 0)$. By using such equation, we can rewrite $[[y_2, y_3], (\!([y_2, y_3])\!)y_2^n y_3^m]$ as $[[y_2, y_3], (\!([y_2, y_3])\!)y_3^m y_2^n]$, which once conjugated by $y_2^{-n} y_3^{-m}$ and inverted becomes

$$[[y_2, y_3], (\!([y_2, y_3])\!)y_2^{-n} y_3^{-m}].$$

This means that the relation $[[y_2, y_3], (\!([y_2, y_3])\!)y_2^n y_3^m]$, assumed to hold for every $(n, m) >_{\text{lex}} (0, 0)$, actually holds for every $(n, m) \neq (0, 0)$.

Now, taking into account the identity $[y_2^{-1}, y_3^{-1}] = (\!([y_2, y_3])\!)y_2 y_3$, from (21) we get

$$(\!([y_2^{-1}, y_3^{-1}])\!)y_3^m y_2^n = (\!([y_2^{-1}, y_3^{-1}])\!)y_2^n y_3^m \tag{22}$$

for every $(n, m) >_{\text{lex}} (0, 0)$, and we can write the commutator $[[y_2^{-1}, y_3^{-1}], (\!([y_2^{-1}, y_3^{-1}])\!)y_2^n y_3^m]$ as $[(\!([y_2, y_3])\!)y_2 y_3, (\!([y_2, y_3])\!)y_2 y_3 y_2^n y_3^m]$. By using equation (21) once again, we immediately have the validity of the relation $[[y_2^{-1}, y_3^{-1}], (\!([y_2^{-1}, y_3^{-1}])\!)y_2^n y_3^m]$ for every $(n, m) >_{\text{lex}} (0, 0)$, and then we can extend such validity to every $(n, m) \neq (0, 0)$, as we did above for the relation $[[y_2, y_3], (\!([y_2, y_3])\!)y_2^n y_3^m]$, but using equation (22) instead of (21).

In conclusion, the presentation in the statement is equivalent to the last presentation of H_τ given in the proof of Theorem 5.1 (after the replacement of y_3 by y_3^{-1}), whose generators $y_2 = x_2 x_1^{-1}$ and $y_3 = (x_3 x_1^{-1})^{-1}$ correspond to the rotations $r_2 r_1$ and $r_1 r_3$, respectively. $\square$

Theorem 5.3. *For a generic triangle τ the group G_τ admits the presentation*

$$\langle x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, [v, (\!(v)\!)(x_2 x_1)^n (x_1 x_3)^m], \, n, m \in \mathbb{Z}, (n, m) >_{\text{lex}} (0, 0)\rangle,$$

with $v = [x_2 x_1, x_1 x_3]$ and the symbols x_1, x_2, x_3 corresponding to r_1, r_2, r_3, respectively.

Proof. We think of G_τ as an extension of H_τ by $\mathbb{Z}_2$ and deduce the presentation of it from that of H_τ given by Theorem 5.2 and the obvious one of $\mathbb{Z}_2$ in the usual way (see [8, Sec. 10.2]).

As the generators we have x_1, corresponding to the reflection r_1 (a lifting to G_τ of the generator of $\mathbb{Z}_2$), and the generators y_2 and y_3 in the presentation of H_τ, corresponding to the rotations $r_2 r_1$ and $r_1 r_3$, respectively.

As the relations, besides the ones in the presentation of H_τ, we have

$$x_1^2, \quad x_1 y_2 x_1^{-1} = y_2^{-1}, \quad x_1 y_3 x_1^{-1} = y_3^{-1}, \tag{23}$$

the first of which comes from the relation of $\mathbb{Z}_2$, while the others express in terms of the generators y_2 and y_3 their conjugates by x_1.

In the new generators x_1, x_2, x_3, with $x_2 = y_2 x_1$ and $x_3 = x_1 y_3$ corresponding to the reflection r_2 and r_3, respectively, the relations (23) reduce to x_1^2, x_2^2, x_3^2, and up to such relations those in the presentation of H_τ read as $[[x_2 x_1, x_1 x_3], (([x_2 x_1, x_1 x_3]))(x_2 x_1)^n (x_1 x_3)^m]$ for every $(n, m) >_{\text{lex}} (0, 0)$. $\square$

Theorem 5.4. *The presentations given in Theorems 5.2 and 5.3 are minimal.*

Proof. We first prove the minimality of the presentation of H_τ, then we see how the argument can be adapted to prove the minimality of the presentation of G_τ. In both cases, the idea is to apply the Reidemeister-Schreier method (see [8, Sec. 9.1]) to obtain a presentation of the subgroup T_τ induced by the presentation of H_τ (resp. G_τ), and show that if any single relation were removed from this last presentation, then the presentation induced on T_τ would give a non commutative group.

We start with the presentation of H_τ in Theorem 5.2, and we choose as a Schreier transversal for $T_\tau \subset H_\tau$ the set $\{y_2^i y_3^j, i, j \in \mathbb{Z}\}$, where $y_2^i y_3^j$ corresponds to the lifting $(r_2 r_1)^i (r_1 r_3)^j \in H_\tau$ of the generic rotation $(s_2 s_1)^i (s_1 s_3)^j \in R_\tau$ in the exact sequence (20).

According to this choice, taking into account that $\lambda_\tau(y_2^i y_3^j y_2) = \lambda_\tau(y_2^{i+1} y_3^j)$, since $R_\tau \subset \mathrm{SO}(2)$ is abelian, and that $y_2^i y_3^j y_3 = y_2^i y_3^{j+1}$, a set of generators for T_τ is

$$\{b_{i,j} = y_2^i y_3^j y_2 y_3^{-j} y_2^{-i-1} = [y_2^i y_3^j, y_2], i, j \in \mathbb{Z}, j \neq 0\}. \tag{24}$$

Moreover, the relations are the transcription in the $b_{i,j}$'s of the words

$$c_{n,m,k,\ell} = [(([y_2, y_3])) y_3^\ell y_2^k, (([y_2, y_3])) y_2^n y_3^m y_3^\ell y_2^k], \tag{25}$$

with $n, m, k, \ell \in \mathbb{Z}$ and $(n, m) >_{\text{lex}} (0, 0)$.

In order to carry out the transcription, we observe that

$$[y_2^i, y_3^j] = \begin{cases} b_{i,j} b_{i+1,j} \ldots b_{-1,j} & \text{if } i \leq 0 \\ (b_{0,j} b_{1,j} \ldots b_{i-1,j})^{-1} & \text{if } i \geq 0 \end{cases}, \tag{26}$$

where we put $b_{i,0} = 1$ for every $i \in \mathbb{Z}$. In fact, apart from the trivial case of $i = 0$, for $i = \pm 1$ we immediately have $[y_2, y_3^j] = b_{0,j}^{-1}$ and $[y_2^{-1}, y_3^j] = b_{-1,j}$. Then, decreasing induction on $i \leq -1$ gives

$$[y_2^i, y_3^j] = y_2^i y_3^j y_2 y_3^{-j} y_2^{-i} y_2^{-1} y_2^{i+1} y_3^j y_2^{-i-1} y_3^{-j} = b_{i,j} [y_2^{i+1}, y_3^j] = b_{i,j} b_{i+1,j} \ldots b_{-1,j},$$

while increasing induction on $i \geq 1$ gives

$$[y_2^i, y_3^j] = y_2 y_2^{i-1} y_3^j y_2^{-1} y_3^{-j} y_2^{-i+1} y_2^{i-1} y_3^j y_2^{-i+1} y_3^{-j} = b_{i-1,j}^{-1} [y_2^{i-1}, y_3^j] = b_{i-1,j}^{-1} b_{i-2,j}^{-1} \ldots b_{0,j}^{-1}.$$

Now, direct inspection shows that

$$(([y_2, y_3])) y_3^\ell y_2^k = [y_2^{-k}, y_3^{-\ell}][y_3^{-\ell}, y_2^{-k+1}][y_2^{-k+1}, y_3^{-\ell+1}][y_3^{-\ell+1}, y_2^{-k}],$$

and after performing the replacements (26), separately for the two cases $k \leq 0$ and $k \geq 1$, we get in both cases

$$(([y_2, y_3])) y_3^\ell y_2^k = b_{-k,-\ell} b_{-k,-\ell+1}^{-1}. \tag{27}$$

Analogously, direct inspection shows that

$$(([y_2, y_3]))y_2^n y_3^m y_2^\ell y_2^k = [y_2^{-k}, y_3^{-m-\ell}][y_3^{-m-\ell}, y_2^{-k-n+1}][y_2^{-k-n+1}, y_3^{-m-\ell+1}]$$
$$\cdot [y_3^{-m-\ell+1}, y_2^{-k-n}][y_2^{-k-n}, y_3^{-m-\ell}][y_3^{-m-\ell}, y_2^{-k}],$$

and after performing the replacements (26), separately for the two cases $k \leq 0$ and $k \geq 1$ if $n = 0$ and for the three cases $k \leq -n$, $-n < k < 0$ and $k \geq 0$ if $n > 0$, we get in all cases

$$(([y_2, y_3]))y_2^n y_3^m y_2^\ell y_2^k = ((b_{-n-k,-m-\ell} b_{-n-k,-m-\ell+1}^{-1})) \prod_{i=0}^{n-1} b_{-n-k+i,-m-\ell} \,. \tag{28}$$

Based on (27) and (28), we have the transcription

$$c_{n,m,k,\ell} = [\, c_{-k,-\ell} c_{-k,-\ell+1}^{-1} \,, ((b_{-n-k,-m-\ell} b_{-n-k,-m-\ell+1}^{-1})) \prod_{i=0}^{n-1} b_{-n-k+i,-m-\ell} \,], \tag{29}$$

for every $n, m, k, \ell \in \mathbb{Z}$ and $(n, m) >_{\mathrm{lex}} (0, 0)$.

At this point, we consider the new set of generators for T_τ

$$\{d_{i,j} = b_{-i,-j} b_{-i,-j+1}^{-1} \,, \ i, j \in \mathbb{Z}\}\,, \tag{30}$$

where $b_{i,0} = 1$ for every $i \in \mathbb{Z}$, and hence $d_{i,1} = b_{-i,-1}$ and $d_{i,0} = b_{-i,1}^{-1}$. Starting from the last two equalities, and proceeding by decreasing induction on $j \leq -1$ and by increasing induction on $j \geq 1$, we obtain

$$b_{i,j} = \begin{cases} d_{-i,-j} d_{-i,-j-1} \ldots d_{-i,1} & \text{if } j \leq 0 \\ (d_{-i,0} d_{-i,-1} \ldots d_{-i,-j+1})^{-1} & \text{if } j \geq 0 \end{cases},$$

with $b_{i,0}$ corresponding to the empty word in the $d_{i,j}$'s in both cases.

Then, by performing these replacements in the equation (29), we obtain as the set of relations for T_τ in the generators $d_{i,j}$

$$\{e_{n,m,k,\ell} = [d_{k,\ell}, ((d_{n+k,m+\ell})) u_{n,m,k,\ell}]\,, \ n, m, k, \ell \in \mathbb{Z}\,, (n, m) >_{\mathrm{lex}} (0, 0)\}\,, \tag{31}$$

where $u_{n,m,k,\ell}$ is a certain word in the $d_{i,j}$'s.

Now, assume by contradiction that a single relation $[[y_2, y_3], (([y_2, y_3]))y_2^{n_0} y_3^{m_0}]$ with $(m_0, n_0) >_{\mathrm{lex}} (0, 0)$ can be removed from the presentation of H_τ, in such a way that we still have a presentation of H_τ. Then, in the set of relations (31) for the induced presentation of T_τ with generators $d_{i,j}$, all the relations $e_{n_0, m_0, k, \ell}$ with $k, \ell \in \mathbb{Z}$ are omitted. This allows us to define a homomorphism $T_\tau \to \Sigma_3$ that sends $d_{0,0}$ and d_{n_0, m_0} to the transpositions $(1\,2)$ and $(2\,3)$, respectively, and any other $d_{i,j}$ to the identity, in contrast with the fact that T_τ is abelian. Indeed, by replacing the generators by the corresponding transpositions in the relation $e_{n,m,k,\ell}$, we always get the identity if at least one of (k, ℓ) and $(n + k, m + \ell)$ does not coincide with $(0, 0)$ or (n_0, m_0). Therefore, since $(n, m), (n_0, m_0) >_{\mathrm{lex}} (0, 0)$, the only possibility for not having the identity is $(k, \ell) = (0, 0)$ and $(n, m) = (n_0, m_0)$. But this cannot happen since the relation $e_{n_0, m_0, 0, 0}$ is missing.

In order to prove the minimality of the presentation of G_τ in Theorem 5.3, we first apply the Reidemeister-Schreier method to derive from it a presentation of the subgroup $H_\tau \subset G_\tau$ with Schreier transversal $\{1, x_1\}$. Arguing as in the proof of Theorem 5.1, with v and $((v))(x_2 x_1)^n (x_1 x_3)^m$ in place of w and $((w))(x_1 x_2)^n (x_1 x_3)^m$, respectively, we obtain for H_τ the presentation

$$\langle y_2, y_3 \mid [[y_2, y_3], (([y_2, y_3]))y_2^n y_3^m], [[y_2^{-1}, y_3^{-1}], (([y_2^{-1}, y_3^{-1}]))y_2^n y_3^m], n, m \in \mathbb{Z}, (n, m) >_{\mathrm{lex}} (0, 0)\rangle.$$

Then, we perform once again the Reidemeister-Schreier method on this presentation to get a presentation of the subgroup $T_\tau \subset H_\tau$ with Schreier transversal $\{y_2^i y_3^j, \, i, j \in \mathbb{Z}\}$. Similar computations as in the first part of this proof leads to a presentation having the same set of generators (30) and relations

$$
\begin{aligned}
e_{n,m,k,\ell} &= [d_{k,\ell}, ((d_{n+k,m+\ell}))u_{n,m,k,\ell}], \\
e'_{n,m,k,\ell} &= [((d_{k+1,\ell+1}))u'_{k,\ell}, ((d_{-n+k+1,-m+\ell+1}))u''_{n,m,k,\ell}],
\end{aligned}
\tag{32}
$$

where $e_{n,m,k,\ell}$ is as in (31), while $u'_{k,\ell}$ and $u''_{n,m,k,\ell}$ are suitable words in the $d_{i,j}$'s.

At this point the minimality of the presentation of G_τ can be deduced by the same argument used above for H_τ, based on the equality between the differences of indices $(k+1, \ell+1) - (-n+k+1, -m+\ell+1) = (n+k, m+\ell) - (k, \ell) = (n, m)$ in (32). $\square$

6. EXAMPLES OF NON-GENERIC RELATIONS

As we have seen in the previous sections, apart from the obvious involutive property of the r_i's, the only generic relations in G_τ, that is the ones holding for τ a generic triangle or equivalently for every triangle τ (by Lemma 2.4), are the commutators of the translations in the free abelian subgroup T_τ generated by the conjugates of t_1.

Here, we briefly discuss the existence of extra non-generic relations for the subgroup T_τ, and hence for the group G_τ, in the case when the triangle τ is typical but not generic. In this respect, typical triangles are expected to present a rich unexplored structure, in some sense complementary to the one encoded by the relations $(r_1 r_2)^{n_3}$, $(r_2 r_3)^{n_1}$ and $(r_3 r_1)^{n_2}$ for a rational triangle τ having angles $m_i \pi / n_i$ with $(m_i, n_i) = 1$, which has been widely considered in the literature after the pioneering work of Coxeter [4].

In Figure 3 two relations of G_τ are represented in terms of the corresponding chain of triangles generated by each next reflection in the word, starting from τ and ending back
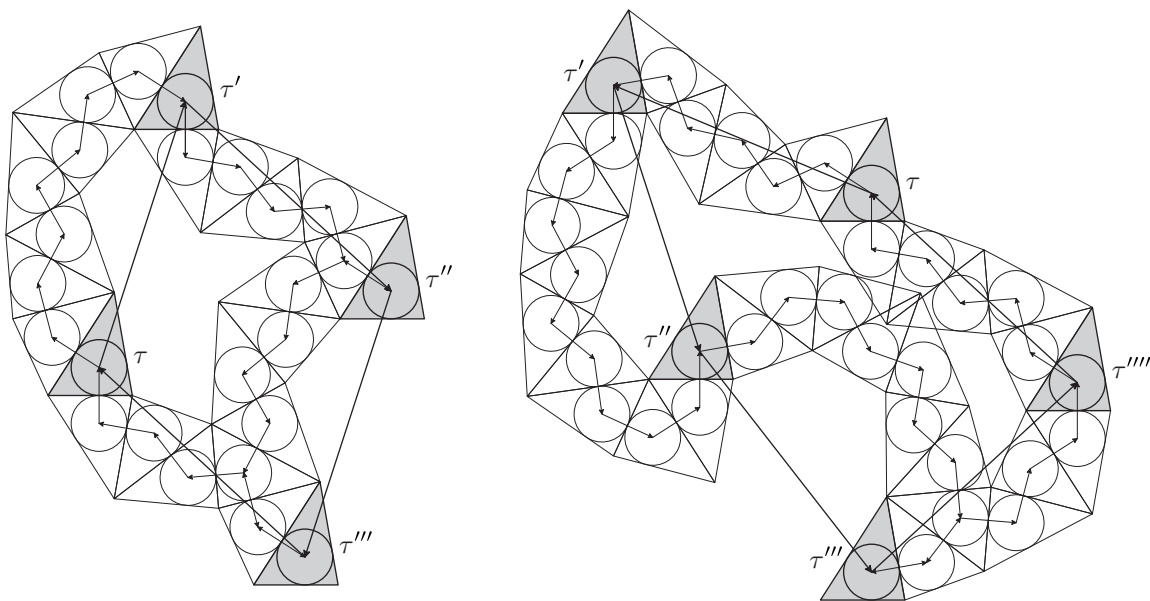

Figure 3. Generic and non-generic relations

to τ. Namely, on the left side there is the generic relation given by the commutator

$$[t_1, ((t_1))r_1 r_3] = t_1 r_3 r_1 t_1 r_1 r_3 t_1^{-1} r_3 r_1 t_1^{-1} r_1 r_3$$
$$= r_1 r_2 r_3 r_1 r_3 r_1 r_2 r_3 r_1 r_2 r_1 r_3 r_2 r_1 r_3 r_1 r_3 r_2 r_1 r_3 r_2 r_3 ,$$

where some r_i^2 has been canceled in the last expression, while on the right side there is the non-generic relation

$$t_1 \cdot ((t_1^{-1}))r_1 \cdot ((t_1))r_1 r_3 r_2 \cdot ((t_1^{-1}))r_1 r_3 r_1 \cdot ((t_1^{-1}))r_3$$
$$= r_1 r_2 r_3 r_1 r_2 r_3 r_1 r_3 r_2 r_1 r_2 r_3 r_1 r_2 r_3 r_1 r_3 r_2 r_1 r_3 r_1 r_3 r_2 r_1 r_3 r_2 r_3 r_1 r_2 r_1 r_3 r_2 r_1 r_3 ,$$

which holds only for the triangles τ whose angles α_i satisfy a specific condition (in particular for all the triangles such that $2\cos(2\alpha_2 + 2\alpha_3) - 2\cos 2\alpha_2 = 1$).

The big vectors superposed to the chains of triangles in the figure correspond to the expression of the relation as a word in the set $C(t_1)$ of generators of the translation subgroup T_τ, while the small vectors indicate the displacement of the incenter of the triangle under the action of each next reflection in the expression of the relation as a word in the generators r_1, r_2 and r_3 of G_τ. The lengths of these two word representations of a relation, in the generators of T_τ and G_τ respectively, provide relatively independent measures of its complexity.

The following table reports the number of stable words in the generators r_1, r_2 and r_3 of G_τ up to length 24, which are cyclically reduced with respect to the cancellation of the r_i^2's, and pairwise distinct up to permutation of indices, inversion, conjugation and commutation of stable words. These have been obtained by a computer procedure in three steps: first, the generation of a complete list of all the cyclically reduced stable words of a given length; then, the elimination of duplicates up to permutation of indices, inversion and cyclic permutations of the word; finally, the detection of the remaining pairs of words (even of different lengths) equivalent up to conjugation and commutation of stable words, by comparing their expressions as linear combinations of vectors in $C(t_1)$.

	1	2	3	4	5	6	7	8	9	10	11	12	Total
6	1	–	–	–	–	–	–	–	–	–	–	–	1
8	–	–	–	–	–	–	–	–	–	–	–	–	0
10	–	1	–	–	–	–	–	–	–	–	–	–	1
12	–	2	1	–	–	–	–	–	–	–	–	–	3
14	–	2	2	1	–	–	–	–	–	–	–	–	5
16	–	2	4	1	1	–	–	–	–	–	–	–	8
18	–	2	8	8	3	3	1	–	–	–	–	–	25
20	–	2	10	24	11	4	3	1	–	–	–	–	55
22	–	4	22	53	40	23	10	9	4	1	–	–	166
24	–	2	23	103	129	87	49	24	15	9	2	1	444

The total number of words of each length from 6 to 24 with respect to the generators r_1, r_2 and r_3 of G_τ in the last column, is subdivided in the previous columns according to the length from 1 to 12 with respect to the set $C(t_1)$ of generators of T_τ.

Now, in order to determine when a cyclically reduced stable word $r_{i_1} r_{i_2} \ldots r_{i_n}$ represents the identity in T_τ, one could directly observe that, according to Lemma 3.3 and equation (6), this happens if and only if

$$\sum_{j=1}^{n}(v_{i_j})s_{i_{j+1}} \ldots s_{i_n} = 0 . \tag{33}$$

Notice that the j-th term of this summation coincides with half the displacement vector of the incenter of the triangle under the action of the j-th reflection in $r_{i_1} r_{i_2} \ldots r_{i_n}$. Taking into account that all these vectors have the same norm, equation (7) could be applied to rewrite equation (33) as a condition on the angles α_2 and α_3 of τ under which $r_{i_1} r_{i_2} \ldots r_{i_n}$ is a relation for T_τ.

A more convenient approach to the same condition on the angles of T_τ is provided by the proof of Theorem 4.1. Once the translation vector corresponding to the word $r_{i_1} r_{i_2} \ldots r_{i_n}$ has been expressed as a linear combination of vectors in $C(t_1)$, equation (33) can be put in the form

$$\sum_{j=1}^m k_j \left(\!\left(t_1 \right)\!\right) r_{i_{j,1}} r_{i_{j,2}} \ldots r_{i_{j,n_j}} = 0 \,.$$

Then, according to equations (10) and (11), we get the equivalent system

$$\begin{cases} \sum_{j=1}^m k_j \cos(m_{j,2}\alpha_2 + m_{j,3}\alpha_3) = 0 \\ \sum_{j=1}^m k_j \sin(m_{j,2}\alpha_2 + m_{j,3}\alpha_3) = 0 \end{cases}, \tag{34}$$

where $m_{j,2}\alpha_2 + m_{j,3}\alpha_3$ is the oriented angle from t_1 to $\left(\!\left(t_1 \right)\!\right) r_{i_{j,1}} r_{i_{j,2}} \ldots r_{i_{j,n_j}}$.

By a systematic computer search among the stable words up to length 24 generated as said above, we found that the shortest words in the r_i's giving non-generic relations for some typical triangle have length 18. Up to permutation of indices, inversion, conjugation and commutation of stable words, there are two of such words of length 18. As discussed in the Examples 6.1 and 6.2 below, both the relations hold for a continuous family of triangles, forming a curve in the space of parameters

$$\mathcal{T} = \{(\alpha_2, \alpha_3) \mid \alpha_2, \alpha_3 > 0 \text{ and } \alpha_2 + \alpha_3 < \pi\} \,,$$

and almost all the triangles in that family are typical. Moreover, the relation presented in Example 6.1 has minimal length also with respect to the set of generators $C(t_1)$. Indeed it is not difficult to see that, apart from commutators, any extra relation holding in T_τ for a typical triangle τ must have length at least 5 in terms of conjugates of t_1.

Example 6.1. Consider the stable word of length 18

$$(r_1 r_2 r_3 r_2 r_3 r_1 r_2 r_1 r_3)^2 = \left(\!\left(t_1 \right)\!\right) r_1 r_3 r_2 r_1 \cdot t_1 \cdot \left(\!\left(t_1 \right)\!\right) r_3 r_1 r_3 r_2 r_3 \cdot \left(\!\left(t_1 \right)\!\right) r_1 r_3 \cdot \left(\!\left(t_1 \right)\!\right) r_3 r_1 r_3 \,.$$

The corresponding translation vector is

$$\begin{aligned} t_1 &+ \left(\!\left(t_1 \right)\!\right) r_1 r_3 + \left(\!\left(t_1 \right)\!\right) r_1 r_3 r_2 r_1 + \left(\!\left(t_1 \right)\!\right) r_3 r_1 r_3 + \left(\!\left(t_1 \right)\!\right) r_3 r_1 r_3 r_2 r_3 \\ &= t_1 + \left(\!\left(t_1 \right)\!\right) r_1 r_3 + \left(\!\left(t_1 \right)\!\right) r_2 r_3 + \left(\!\left(t_1 \right)\!\right) r_1 r_2 r_1 r_3 + \left(\!\left(t_1 \right)\!\right) r_1 r_3 r_1 r_3 \,, \end{aligned} \tag{35}$$

where the simplification is based on the equation (8), the commutativity of the rotations $s_i s_j$ and $s_k s_l$, and the identities $\left(\!\left(t_1 \right)\!\right) r_1 r_2 r_3 = t_1$ and $s_i^2 = 1$. Now, the oriented angles from t_1 to the five vectors in (35) are respectively given by $0, 2(\alpha_2 + \alpha_3), 2\alpha_2, -2(\alpha_2 - \alpha_3)$ and $4\alpha_2$, and by replacing in (34) we get the system

$$\begin{cases} \cos 2(\alpha_2 + \alpha_3) + \cos 2\alpha_2 + \cos 2(\alpha_2 - \alpha_3) + \cos 4\alpha_2 = -1 \\ \sin 2(\alpha_2 + \alpha_3) + \sin 2\alpha_2 - \sin 2(\alpha_2 - \alpha_3) + \sin 4\alpha_2 = 0 \end{cases} \,.$$

By standard trigonometric identities, this system is equivalent to

$$\begin{cases} (1 + 2\cos 2\alpha_2 + 2\cos 2\alpha_3)\cos 2\alpha_2 = 0 \\ (1 + 2\cos 2\alpha_2 + 2\cos 2\alpha_3)\sin 2\alpha_2 = 0 \end{cases},$$

hence to the equation

$$1 + 2\cos 2\alpha_2 + 2\cos 2\alpha_3 = 0.$$

The curve solutions of this equation in the space of parameters $\mathcal{T}$ is plotted on the left side of Figure 4. Since non-typical triangles form a dense countable union of straight lines in $\mathcal{T}$, it is clear that only countably many triangles along the curve are non-typical. A very special case is represented by the triangle d in the figure, whose angles are all rational multiples of π. Hence, we can conclude that the considered word is a relation in T_τ for uncountably many typical non-generic triangles τ, which form a dense subset of the curve. A sample of them is given by the five triangles a, b, c, e, f depicted in the figure.
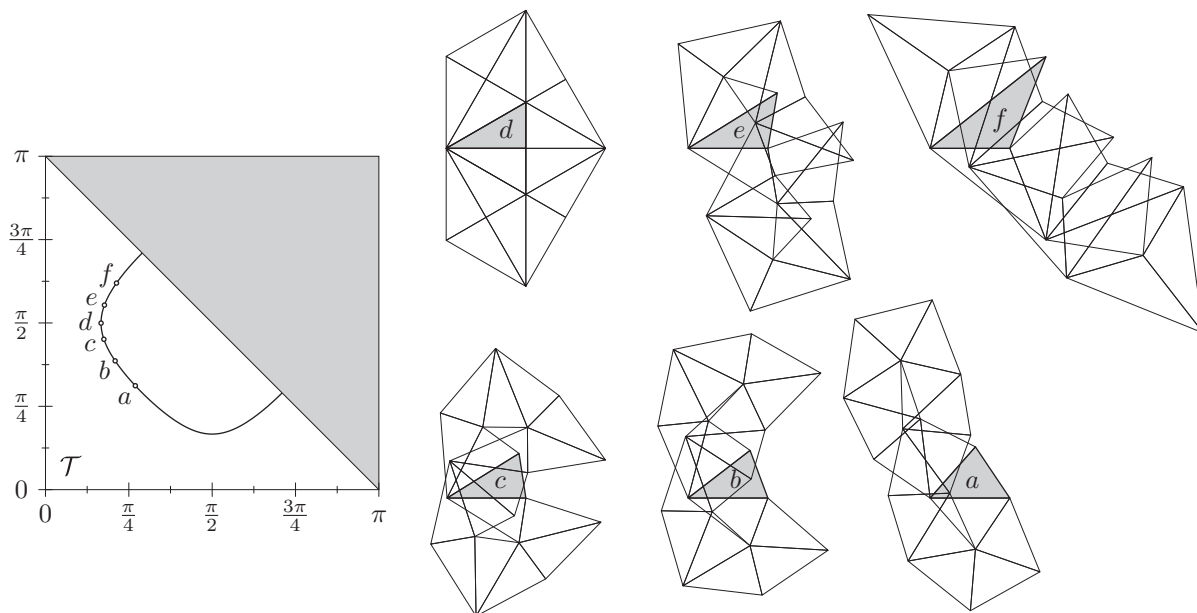

Figure 4. The non-generic relation of Example 6.1

Example 6.2. Arguing as above, we see that the stable word of length 18

$$(r_1 r_2 r_3 r_2 r_3 r_1 r_3 r_1 r_2)^2$$
$$= (\!(t_1)\!)r_1 r_3 r_2 r_1 \cdot t_1 \cdot (\!(t_1)\!)r_1 r_3 r_2 r_3 \cdot (\!(t_1)\!)r_3 r_1 r_3 r_2 r_3 \cdot (\!(t_1)\!)r_1 r_3 \cdot (\!(t_1)\!)r_3 \cdot (\!(t_1)\!)r_1 r_3 r_1 r_2$$

corresponds to the translation vector

$$t_1 + (\!(t_1)\!)r_3 + (\!(t_1)\!)r_1 r_3 + (\!(t_1)\!)r_1 r_3 r_2 r_1 + (\!(t_1)\!)r_1 r_3 r_1 r_2 + (\!(t_1)\!)r_1 r_3 r_2 r_3 + (\!(t_1)\!)r_3 r_1 r_3 r_2 r_3$$
$$= t_1 + (\!(t_1)\!)r_1 r_2 + (\!(t_1)\!)r_1 r_3 + (\!(t_1)\!)r_2 r_3 + (\!(t_1)\!)r_1 r_3 r_1 r_2 + (\!(t_1)\!)r_1 r_3 r_2 r_3 + (\!(t_1)\!)r_1 r_3 r_1 r_3.$$

This leads to the system

$$\begin{cases} (1 + 2\cos 2\alpha_2 + 2\cos 2\alpha_3 + 2\cos 2(\alpha_2 + \alpha_3))\cos 2\alpha_2 = 0 \\ (1 + 2\cos 2\alpha_2 + 2\cos 2\alpha_3 + 2\cos 2(\alpha_2 + \alpha_3))\sin 2\alpha_2 = 0 \end{cases},$$

hence to the equation

$$1 + 2\cos 2\alpha_2 + 2\cos 2\alpha_3 + 2\cos 2(\alpha_2 + \alpha_3) = 0\,.$$

Hence, also in this case we can conclude that the considered word is a relation in T_τ for uncountably many typical non-generic triangles τ, which form a dense subset of the curve represented by the equation.

Besides the two relations of length 18 given in the previous examples, our computer search also detected other non-generic relations holding for all the triangles along a curve in the parameter space $\mathcal{T}$, hence for uncountably many typical triangles. Namely, there are 6 such relations of length 22 and 5 of length 24, but none of length 20.

Moreover, we found a certain number of non-generic relations holding only for isolated typical triangles. One of such relations is discussed in Example 6.3.

Actually, systematic search produced even shorter relations holding in isolated triangles, which present strong evidence of being typical. But we were not able to prove that such triangles are really typical. The shortest one has length 22, and it is the unique one of that length, up to permutation of indices, inversion, conjugation and commutation of stable words. Up to the same equivalence, there are also 20 similar relations of length 24, some of which have the minimal length 5 with respect to $C(t_1)$. Such further relations in conjecturally typical triangles are illustrated by Example 6.4.

Example 6.3. Consider the stable word of length 32

$$r_1 r_3 r_1 r_2 r_3 r_2 r_3 r_1 r_2 r_3 r_2 r_3 r_1 r_2 r_3 r_1 r_2 r_3 r_1 r_2 r_3 r_2 r_3 r_1 r_3 r_1 r_2 r_1 r_2 r_3 r_1 r_3$$
$$= (\!(t_1)\!) r_3 r_1 \cdot (\!(t_1)\!) r_2 r_1 \cdot (\!(t_1^2)\!) r_3 r_1 \cdot (\!(t_1)\!) r_2 r_1 \cdot (\!(t_1)\!) r_1 r_3 r_2 r_1 \cdot t_1 \cdot (\!(t_1)\!) r_1 r_3\,,$$

whose corresponding translation vector is

$$t_1 + (\!(t_1)\!) r_1 r_3 + 2(\!(t_1)\!) r_2 r_1 + 3(\!(t_1)\!) r_3 r_1 + (\!(t_1)\!) r_2 r_3\,.$$

Proceeding as in the previous examples, we see that this word represents the identity in T_τ if and only if the angles α_2 and α_3 satisfy the system

$$\begin{cases} 4\cos 2\alpha_2 + 2\cos 2\alpha_3 + \cos 2(\alpha_2 + \alpha_3) = -1 \\ 2\cos(\alpha_2 + \alpha_3)\big(\sin(\alpha_2 + \alpha_3) - 2\sin(\alpha_2 - \alpha_3)\big) = 0 \end{cases}.$$

Apart from the rational (mod π) solution $\alpha_2 = \alpha_3 = \pi/4$, the only other acceptable solution is

$$\begin{cases} \alpha_2 = \arctan\sqrt{2} \\ \alpha_3 = \arctan\dfrac{\sqrt{2}}{3} \end{cases}.$$

According to Theorem 2 of [3], this solution can be written in the form

$$\begin{cases} \alpha_2 = q\pi \pm \langle 3\rangle_2 \\ \alpha_3 = q'\pi \pm \langle 11\rangle_2 \end{cases},$$

for certain rational numbers q and q', and certain angles $\langle 3\rangle_2$ and $\langle 11\rangle_2$ that are rationally independent together with π. This implies that the triangle is typical. The chains of reflections realizing the relation for such triangle is shown in Figure 5.
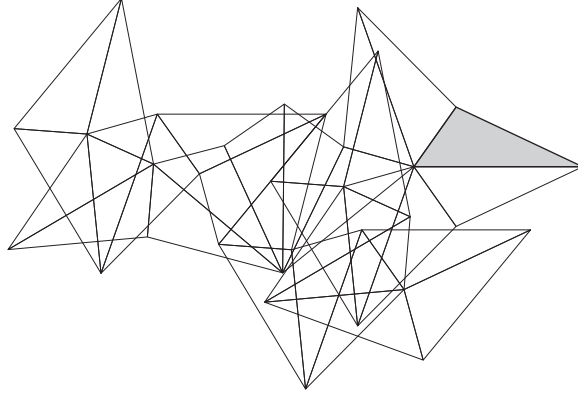
Figure 5. The non-generic relation of Example 6.3

Example 6.4. The stable word of length 22

$$r_1r_2r_1r_2r_3r_1r_3r_1r_3r_2r_3r_1r_2r_3r_2r_3r_2r_3r_2r_3r_1r_2$$
$$= (\!(t_1)\!)r_2r_1 \cdot (\!(t_1)\!)r_1r_3r_2r_1 \cdot t_1 \cdot (\!(t_1)\!)r_1r_3 \cdot (\!(t_1)\!)r_2r_3r_1r_3 \cdot (\!(t_1)\!)r_2r_3r_2r_3r_1r_3 \cdot (\!(t_1)\!)r_3$$

represents the identity in T_τ if and only if

$$\begin{cases} \cos 2\alpha_2 + 2\cos 2\alpha_3 + \cos 2(\alpha_2 + \alpha_3) + \cos 2(2\alpha_2 + \alpha_3)\cos 2(3\alpha_2 + 2\alpha_3) = -1 \\ \sin 2\alpha_2 + \sin 2(\alpha_2 + \alpha_3) + \sin 2(2\alpha_2 + \alpha_3)\sin 2(3\alpha_2 + 2\alpha_3) = 0 \end{cases}.$$

The only two acceptable approximate solutions of the system are

$$\begin{cases} \alpha_2 = 0.3675592642\,\pi \\ \alpha_3 = 0.1932064551\,\pi \end{cases} \text{and} \quad \begin{cases} \alpha_2 = 0.5971477967\,\pi \\ \alpha_3 = 0.2299624978\,\pi \end{cases}.$$

Analogously, the stable word of length 24

$$r_1r_2r_1r_2r_1r_3r_1r_3r_2r_1r_3r_1r_2r_1r_3r_2r_3r_2r_3r_1r_2r_3r_2r_3$$
$$= (\!(t_1^{-1})\!)r_3r_1 \cdot (\!(t_1^{-1})\!)r_2r_1r_3r_1 \cdot (\!(t_1^{-1})\!)r_3r_1r_2r_1r_3r_1 \cdot t_1 \cdot (\!(t_1)\!)r_2r_3$$

represents the identity in T_τ if and only if

$$\begin{cases} \cos 2\alpha_2 - \cos 2(\alpha_2 + \alpha_3) + \cos 2(\alpha_2 - \alpha_3) + \cos 2(2\alpha_2 - \alpha_3) = 1 \\ \sin 2\alpha_2 + \sin 2(\alpha_2 + \alpha_3) + \sin 2(\alpha_2 - \alpha_3)\sin 2(2\alpha_2 - \alpha_3) = 0 \end{cases}.$$

The only acceptable approximate solution of the system is

$$\begin{cases} \alpha_2 = 0.2961623095\,\pi \\ \alpha_3 = 0.4392394514\,\pi \end{cases}.$$

The chains of reflections realizing both the non-generic relations above are shown in Figure 6, on the left side for the two triangles where the former relation holds and on the right side for the unique triangle where the latter holds.
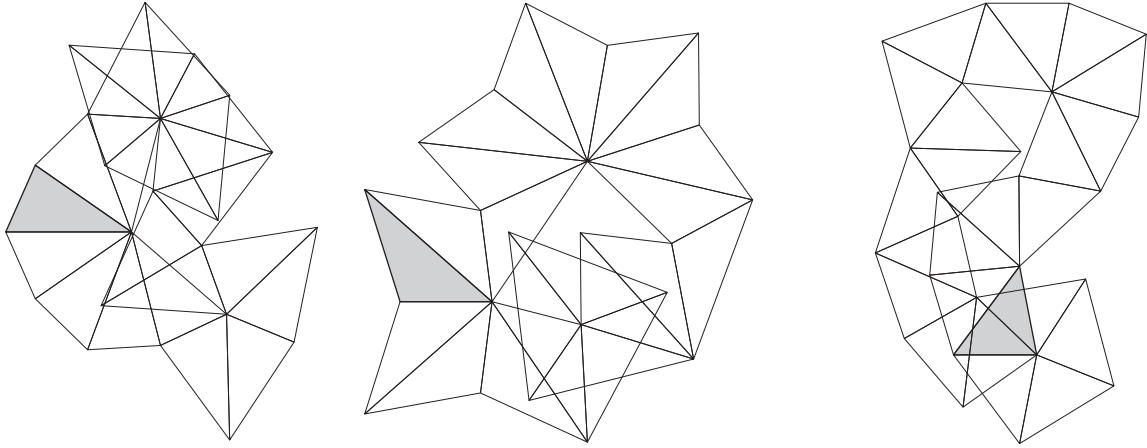
Figure 6. The non-generic relations of Example 6.4

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