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A UNIVERSAL RIBBON SURFACE IN B^4

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Abstract

We construct an orientable ribbon surface $F \subset B^4$, which is universal in the following sense: any orientable 4-manifold $M \cong B^4 \cup$ 1-handles $\cup$ 2-handles can be represented as a cover of B^4 branched over F.

Keywords: 4-manifold, branched covering, universal surface.

AMS Classification: 57M12, 57N13, 57Q45.

Introduction

In the early seventies H.M. Hilden, U. Hirsch and J.M. Montesinos independently proved that any closed orientable 3-manifold can be represented as a 3-fold simple covering of S^3 branched over a knot (cf. [5], [11] and [15]).

Ten years later, W. Thurston constructed the first universal link. He called a link $L \subset S^3$ universal iff for any closed orientable 3-manifold there exists an n-fold (in general non-simple) covering $M \to S^3$ branched over L. Subsequently, other universal links and knots were constructed by H.M. Hilden, M.T. Lozano, J.M. Montesinos and W.C. Whitten. The basic idea of these constructions is the following: symmetrize the branching links given by the Hilden-Hirsch-Montesinos representation theorem, making them sublinks of the preimage of a fixed link with respect to a fixed branched covering $S^3 \to S^3$. (cf. [23], [6], [7], [8], [9] and [17]).

More recently, M. Iori and R. Piergallini obtained a representation theorem of closed orientable smooth 4-manifolds as 5-fold simple covering of S^4 branched over a smooth surface (cf. [20] and [12]). Then, it makes sense to look for a universal surface in S^4, satisfying a universal property analogous to that one of a universal link in the 3-dimensional case. But unfortunately, the symmetrization technique used for branching links in S^3 seems unlikely to be directly adaptable to branching surfaces in S^4.

In this paper, we show how certain ribbon branching surfaces in B^4 can be symmetrized, in order to get a universal orientable ribbon surface, for representing any compact bounded orientable 4-manifold $M \cong B^4 \cup$ 1-handles $\cup$ 2-handles as a branched cover of B^4. Such 4-manifolds turn out to be relevant for the presentation of all the closed orientable smooth 4-manifolds, making no difference how 3- and 4-handles are attached to them (cf. [13]). Hence, our result could be also useful in constructing a universal surface in S^4. Namely, we prove the following theorem.

THEOREM. *There exists an orientable ribbon surface $F \subset B^4$, such that any compact orientable 4-manifold $M \cong B^4 \cup$ 1-handles $\cup$ 2-handles is a cover of B^4 branched over F.*

We recall that a smooth compact surface $F \subset B^4$ with $\mathrm{Bd}\, F \subset S^3$ is called a *ribbon surface* if the Euclidean norm restricts to a Morse function on F with no local maxima in $\mathrm{Int}\, F$. Assuming $F \subset R^4_+ \subset R^4_+ \cup \{\infty\} \cong B^4$, we note that this property is topologically equivalent to the fact that the fourth Cartesian coordinate restricts to a Morse function on F with no local minima in $\mathrm{Int}\, F$. Such a surface $F \subset R^4_+$ can be isotoped to make its orthogonal projection $F_0 \subset R^3$ a self-transversal immersed surface, whose double points form disjoint arcs as in Figure 1. Here, as well as in the following figures, we shade the surface roughly according to the fourth coordinate.

FIGURE 1.

We will refer to F_0 as a 3-dimensional *diagram* of F. It is worth observing that any immersed compact surface $F_0 \subset R^3$ with no closed components, all self-intersections of which are as above, is the diagram of a ribbon surface F uniquely determined up to isotopy. In fact, F can be obtained by pushing $\mathrm{Int}\, F_0$ inside $\mathrm{Int}\, B^4$, in such a way that all the self-intersections disappear.

We also recall that a smooth map $p : M \to B^4$ is called a *branched covering* with *branching surface* $F \subset B^4$, if the restriction $p_| : M - p^{-1}(F) \to B^4 - F$ is a regular (as a smooth map) ordinary covering of finite degree d. Assuming F is minimal with respect to this property, we have $F = p(S)$ where $S \subset M$ is the *singular surface* of p. For each $x \in S$, there exists a neighborhood U of x in M, such that the restriction $p_| : U \to p(U)$ is topological equivalent to the complex map defined by $(z_1, z_2) \mapsto (z_1^{d_x}, z_2)$ (with $\mathrm{Im}\, z_2 \geq 0$ if $x \in \mathrm{Bd}\, M$), where d_x is a positive integer called *local degree* or *branching index* of p at x. We say that p is *simple* if the restriction $p_| : S \to F$ is injective and $d_x = 2$ for any $x \in S$.

Since p is uniquely determined (up to diffeomorphism) by its restriction over $B^4 - F$, we can describe it in terms of its *monodromy* $\pi_1(B^4 - F) \to \Sigma_d$ (defined up to conjugation in Σ_d, depending on the numbering of the sheets), that is, by giving the monodromies of any set of meridian loops around S generating $\pi_1(B^4 - F)$. Usually, this is done by labelling a 3-dimensional diagram of F with the permutations corresponding to the standard Wirtinger's generators of $\pi_1(B^4 - F)$, which are assumed to cross the diagram from back to front. From this perspective, p is simple if and only if all the labels are transpositions.

The paper is entirely devoted to proving the theorem above. In particular, the symmetrization procedure is described in Section 3 and the universal surface F is depicted in Figure 27. Sections 2 and 1 are respectively aimed to show that any 4-manifold M as in the statement is a simple covering of B^4 branched over a suitable ribbon surface and to introduce the covering moves needed for symmetrizing such a ribbon branching surface.

1. Some covering moves

By a *covering move*, we mean any modification on a labelled surface determining a branched covering $p : M \to B^4$, that preserves the covering manifold M up to diffeomorphism. All the covering moves considered in this paper are *local*, that is the modification takes place inside a cell and can be performed independently of the form of the rest of labelled branching surface outside. In the figures describing these moves, we will draw only the part of the labelled branching surface inside the relevant cell, assuming everything else to be fixed.

Of course, the notion of covering move makes sense for coverings between PL manifolds of any dimension m branched over arbitrary $(m-2)$-dimensional subcomplexes of the range. Before defining our moves, we roughly state two very general equivalence principles in this broader context and discuss some applications to our specific situation. Several special cases of these principles already appeared in the literature and we can think of them as belonging to the "folklore" of branched coverings.

Disjoint monodromies crossing. *Subcomplexes of the branching set of a covering that are labelled with disjoint permutations can be isotoped independently from each other without changing the covering manifold.*

The reason why this principle holds is quite simple. Namely, being the labelling of the considered subcomplexes disjoint, the sheets non-trivially involved by them do not interact, at least locally over the region where the isotopy takes place. Hence, relative position of such subcomplexes is not relevant in determining the covering manifold.

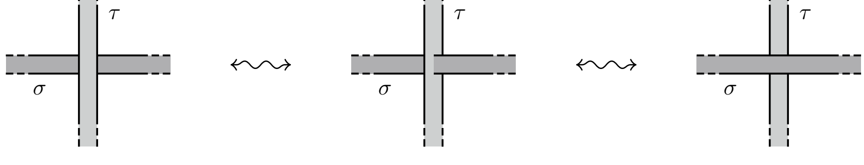


Figure 2.

In particular, this principle allows crossing changes in diagrams when the involved monodromies are disjoint. For example, this is the case of one of the well-known Montesinos moves (cf. [18], [20], [1] or [3]) for simple coverings of S^3 branched over a link. Such crossing change has already been used in the construction of universal links (cf. [7] and [9]). In the same spirit, we specialize the above principle in our 4-dimensional context, by considering the crossing change move described in Figure 2, where $\sigma, \tau \in \Sigma_d$ are arbitrary disjoint permutations.

It is worth observing that, abandoning transversality, the disjoint monodromies crossing principle also gives the special case of the next one when the σ_i's are disjoint and L is empty.

Coherent monodromies merging. *Let $p : M \to N$ be any branched covering with branching set B_p and let $\pi : E \to K$ be a connected disk bundle embedded in N, in such a way that: 1) there exists a (possibly empty) subcomplex $L \subset K$ for which $B_p \cap \pi^{-1}(L) = L$ and the restriction of π to $B_p \cap \pi^{-1}(K - L)$ is an unbranched covering of $K - L$; 2) the monodromies $\sigma_1, \dots, \sigma_n$ relative to a fundamental system $\omega_1, \dots, \omega_n$ for the restriction of p over a given disk $D = \pi^{-1}(x)$, with $x \in K - L$, are coherent in the sense that $p^{-1}(D)$ is a disjoint union of disks. Then, by contracting the bundle E fiberwise to K, we get a new branched covering $p' : M \to N$, whose branching set $B_{p'}$ is equivalent to B_p, except for the replacement of $B_p \cap \pi^{-1}(K - L)$ by $K - L$, with the labelling uniquely defined by letting the monodromy of the meridian $\omega = \omega_1 \dots \omega_n$ be $\sigma = \sigma_1 \dots \sigma_n$.*

We remark that, by connection and property 1), the coherence condition required in 2) actually holds for any $x \in K$. Then, we can prove that p and p' have the same covering manifold, by a straightforward fiberwise application of the Alexander's trick to the components of the bundle $\pi \circ p : p^{-1}(E) \to K$. A coherence criterion can be immediately derived from Section 1 of [19].

We will mainly apply the merging principle to "parallel" components of the branching surface with coherent monodromies, in order to control the number of such components (cf. the below discussion of stabilization and Figures 6, 8, 9, 11).

However, this principle originated from a classical perturbation argument in algebraic geometry and appeared in the literature as a way to deform non-simple coverings between surfaces into simple ones, by going in the opposite direction from p' to p (cf. [2]). In dimension 3, it can be used in this direction, not only for achieving simplicity (cf. [3] or [4]), but also for removing singularities from the branching set (cf. [3]). Moreover, it has been used in the construction of universal links, for controlling the branching indices (cf. [9]).

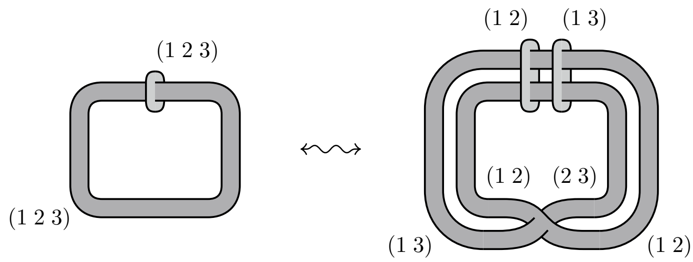


Figure 3.

Figure 3 shows an example of an application of the merging principle to coverings of B^4 branched over ribbon surfaces. Here, the absolute version (with $L = \emptyset$) of the principle is applied in turn to both the components of the branching surface on the left side (letting K be a component and $\pi : E \to K$ be its normal bundle). There is no obstruction to generalize this example, to show that any covering of B^4 branched over a ribbon surface can be deformed into a simple one. For applications of the relative version of the principle (in both directions) see Figures 6 and 9.

Now, we pass to define our moves on labelled ribbon surfaces representing branched coverings of B^4. Concerning the assumptions on the monodromies, the definitions are given on a level of generality which is not the highest possible, but is still higher than needed for our present purposes. We made this choice because such moves are interesting in their own right. In the next section we will use only stabilization and Moves 3 and 4. Moves 1 and 2 are used here to get the others. Let us start with some considerations about the well-known notion of stabilization.

Stabilization. The basic version consists in the addition of an extra trivial sheet, the $(d+1)$-th, to a given d-fold branched covering. In terms of branching surface, this means adding a separate trivial disk with label $(i\ d{+}1)$, where $1 \leq i \leq d$. Now, we can iterate this process l times, by adding l trivial disks with labels $(i_1\ d{+}1), \dots, (i_l\ d{+}l)$, where $1 \leq i_j \leq d + j - 1$. Of course, we can assume the disks to be parallel and it is easy to realize that their monodromies are coherent, whatever i_j's we choose. Hence, we can merge all the disks into one. In particular, if all the i_j's are distinct, the label of this disk is given by the product of l disjoint transposition $(i_1\ d{+}1) \dots (i_l\ d{+}l)$. We will refer to the addition of such a labelled disk as the *multi-stabilization* involving the sheets $i_1, \dots, i_l$.

Move 1. This move is described in Figure 4, where $j_1, \dots, j_l$ and $k_1, \dots, k_l$ are assumed to be all distinct (cf. [9] and [12] for the case of $l = 1$). It can be obtained

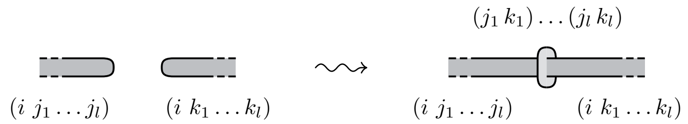


FIGURE 4.

by a straightforward application of the main technique of [12], that is by extending the covering in the left side of the figure to certain cancelling 1- and 2- handles added to M and B^4, in such a way that the branching surface becomes as in the right side.

Namely, we add to B^4 a 1-handle H^1, connecting two small 3-balls around the tips of the tongues in the left side of the figure, and then a 2-handle H^2 complementary to H^1, whose attaching loop λ meets B^4 along a horizontal line avoiding the tongues. The covering instructions can be extended to these handles, by assigning to λ the monodromy $(j_1\ k_1) \dots (j_l\ k_l)$ and by completing the branching surface with the cocore disk of H^2 labelled with the same monodromy of λ. After cancelling H^1 and H^2, the new branching surface and monodromy look like that in the right side of Figure 4. We leave to the reader to check that, in the new covering manifold, there are $d - l$ 1-handles over H^1 and the same number of 2-handle over H^2 and that they cancel (non-trivially) to give back M again.

We remark that Move 1 could also be derived from the special case when $l = 1$, with an inductive argument analogous to the one used below for Move 3.

Move 2. Our second move is given by Figure 5. Here, the σ in the left side is any permutation in Σ_d, while the σ in the right side is the same permutation thought

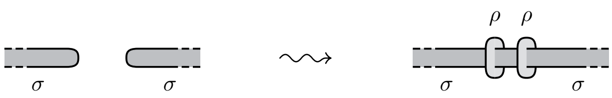


FIGURE 5.

in $\Sigma_{d'}$, for a certain $d' > d$, and $\rho \in \Sigma_{d'}$ is a product of disjoint transpositions which depends on σ. Differently from the previous one, this move changes the degree of the covering. In fact, we can transform the left side of Figure 5 into the right one, by performing a suitable multi-stabilization followed by a generalized version of our first move. Let $\sigma = \gamma_1 \dots \gamma_h$ a decomposition of the given permutation σ into disjoint cycles. For the sake of exposition, we proceed by induction on h.

If $h = 1$ we can write $\sigma = (i\ j_1\ \dots\ j_l)$. In this case, we perform on the covering represented by the diagram on the left side of Figure 5 a multi-stabilization involving the sheets $j_1, \dots, j_l$. As a result, one trivial disk with label $\rho = (j_1\ d{+}1) \dots (j_l\ d{+}l)$ appears in the diagram. We stretch one of the two tongues to pass through such disk, so that its monodromy beyond becomes $\sigma^\rho = \rho^{-1}\sigma\rho = (i\ d{+}1\ \dots\ d{+}l)$. At this point, Move 1 immediately gives the diagram on the right side of the figure.

The case of $h > 1$ can be reduced to the inductive hypothesis by means of crossing changes and the merging principle, as shown in Figure 6. Here, $\sigma' = \gamma_1 \dots \gamma_{h-1}$, $\sigma'' = \gamma_h$, ρ' (resp. ρ'') is the product of disjoint transpositions resulting from applications of Move 1 to the tongues labelled with σ' (resp. σ''), and $\rho = \rho'\rho''$. Starting

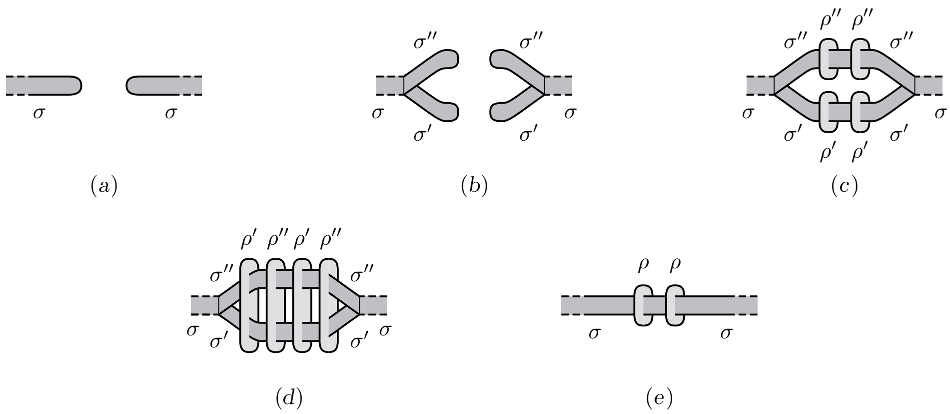


FIGURE 6.

from (a), we apply the following, in sequence: the merging principle to get (b); the inductive hypothesis to get (c); crossing changes to get (d); the merging principle again to get (e).

Move 3. Our third move is the one of Figure 7, where the permutations σ and ρ, as well as covering degrees, are the same as those of Figure 5. We can limit ourselves

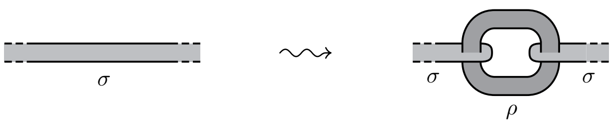


FIGURE 7.

to considering the case when σ is a cycle, since the general case can be derived by induction on the length of a cyclic decomposition of σ, with the same argument as that used for Move 2 (think of Figure 9 below, as if it were labelled analogously to Figure 6).

So, we assume $\sigma = (i\ j_1 \dots j_l)$ and proceed by induction on l. Figure 8 shows how to deal with the case of $l = 1$, when $\sigma = (i\ j_1)$ and $\rho = (j_1\ d{+}1)$.

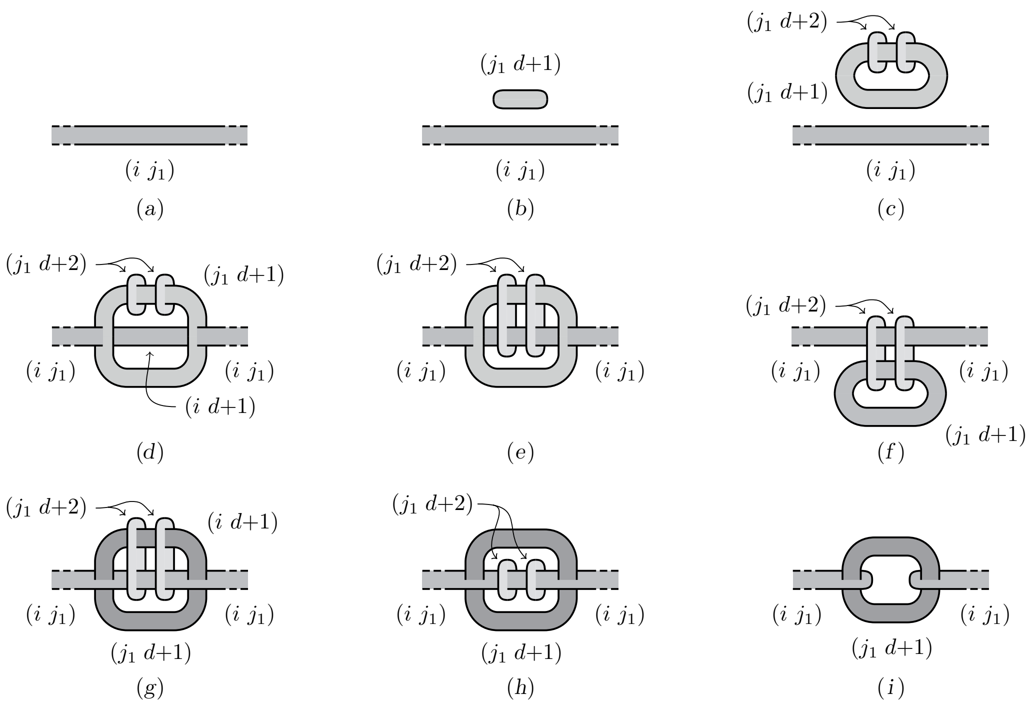


FIGURE 8.

We observe that diagrams (c) and (d) represent isotopic surfaces, and same holds for diagrams (e), (f) and (g). Moreover: (b) is a stabilization of (a); (c) and (i) are obtained from the previous diagrams by Move 2 and its inverse; (e) and (h) by

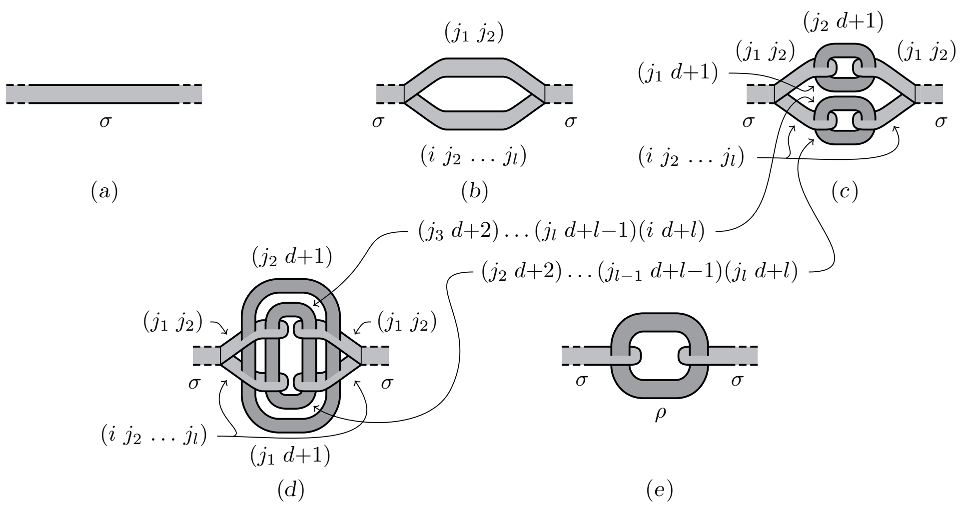


FIGURE 9.

crossing changes. The inductive step is described in Figure 9. Here, the sequence of operations needed to get the various diagrams is the same of Figure 6.

Move 4. Differently from the previous ones, this move is defined only for simple monodromies, but it does not preserve simplicity. It is depicted in Figure 10, where $\tau_1, \tau_2 \in \Sigma_d$ are arbitrary distinct transpositions and $\tau_3 = \tau_1^{\tau_2} = \tau_2^{-1}\tau_1\tau_2$, while each ρ_j is a product of two disjoint transpositions which depend on the τ_i.

FIGURE 10.

The following Figure 11 tells us why this is a true covering move if τ_1 and τ_2 are not disjoint. Here, (b) and (c) are obtained by Move 3 (followed by isotopy in the former step), (d) by crossing change, and (e) by merging principle. We leave to the reader to adapt the monodromies of Figure 11 to the easier case of τ_1 and τ_2 disjoint.

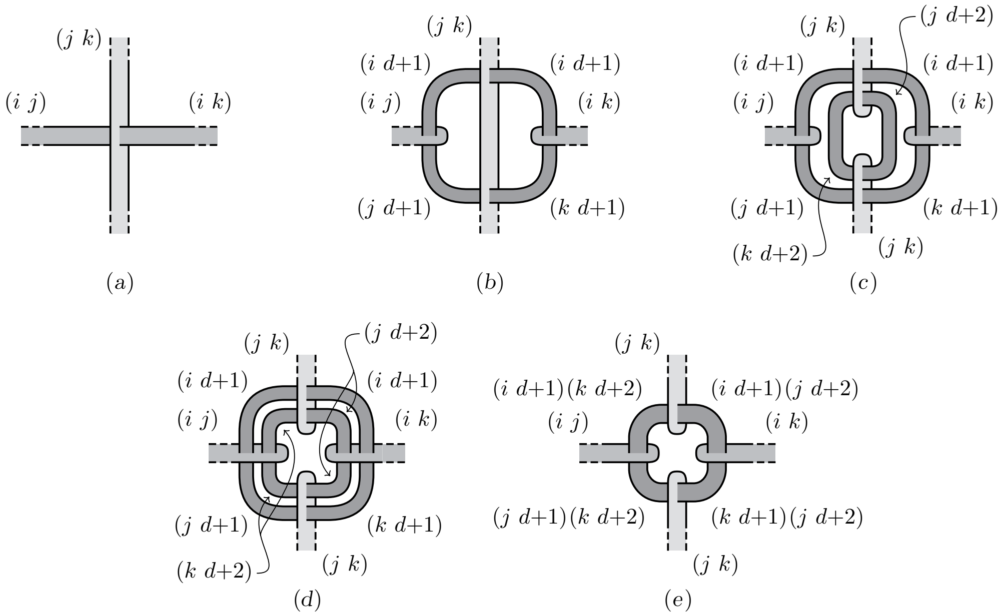


FIGURE 11.

2. Special covering presentations

Given a compact bounded orientable 4-manifold M as in the statement of our theorem, that is $M \cong B^4 \cup$ 1-handles $\cup$ 2-handles, we want to present it as a simple covering of B^4 branched over a suitable orientable ribbon surface.

By a result of Montesinos [16], we know that M is a 3-fold simple cover of B^4 branched over a possibly non-orientable ribbon surface $F \subset B^4$. A variation of the Montesinos's argument actually shows that F can be chosen orientable. Alternatively, we can think of M as a topological Lefschetz fibration over B^2 and represent it as 3-fold simple cover of B^4 branched over a braided surface (cf. [14, Remark 3]).

However, we will construct a special covering presentation of M by a technique similar to one used in [14]. This choice, renouncing control of the degree of the covering, which is not relevant in this context, will eventually allows us to get a simpler universal surface. Nevertheless, it is worth observing that the symmetrization process described in the next section could be arranged to work starting from a generic ribbon branched surface.

Let the generic Kirby diagram of Figure 12 represent M. Here, as well as in Figure 13, the framings are assumed to coincide with the blackboard ones outside the box.

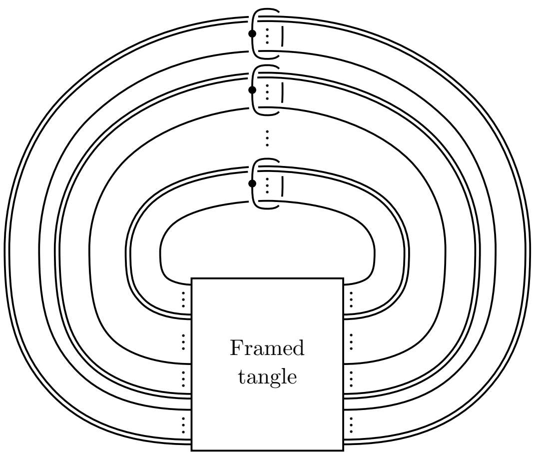


FIGURE 12.

By the classical Alexander's argument, we can modify this diagram in order to make the framed tangle inside the box into a framed braid. Moreover, by inserting a certain number of kinks and enlarging them to form new braid strings, we can assume that all the framings coincide with the blackboard ones. Figure 13 shows the resulting diagram cut open in the upper part, after the 1-handles have been isotoped to the lower part.

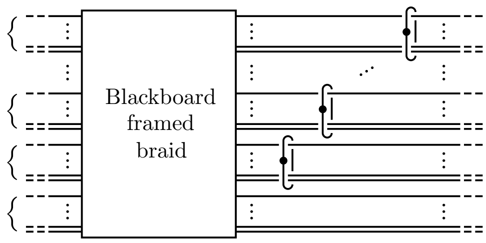


FIGURE 13.

The rest of this section is devoted to showing how the handle presentation of Figure 13 can be converted into a simple covering $M \to B^4$ branched over an orientable ribbon surface. We need first to specify some more details of such handle presentation. Let m and n be respectively the number of 1-handles and 2-handles. We denote by $K^1_1, \dots, K^1_m$ the vertical trivial loops representing the 1-handles in the diagram, indexed from left to right, and by $K^2_1, \dots, K^2_n$ the braid components forming the attaching loops of the 2-handles, indexed according to their lowermost occurrence on the left, from bottom to top. We assume the K^2_i counter-clockwise oriented. For any $i = 1, \dots, n$, we call s_i the number of strings of K^2_i and we put $t_i = s_1 + \dots + s_i$. As a notational convenience, we also put $t_0 = 0$. Moreover, H^i_j will indicate the i-handle corresponding to K^i_j.

To begin with, we consider the simple branched covering of B^4 with $t_n + m + 1$ sheets numbered from 0 to $t_n + m$, whose branching surface consists of the trivial family of disjoint disks $D_1, \dots, D_{t_n+2m} \subset B^4$ and whose monodromy is given as follows: the disks $D_{t_{i-1}+1}, D_{t_{i-1}+2}, \dots, D_{t_i}$, that will be used for the 2-handle H^2_i, have respective monodromies $(0\ t_{i-1}{+}1), (t_{i-1}{+}1\ \ t_{i-1}{+}2), \dots, (t_i{-}1\ \ t_i)$; the disks D_{t_n+2j-1} and D_{t_n+2j}, corresponding to the 1-handle H^1_j, have the same monodromy $(0\ t_n{+}j)$.

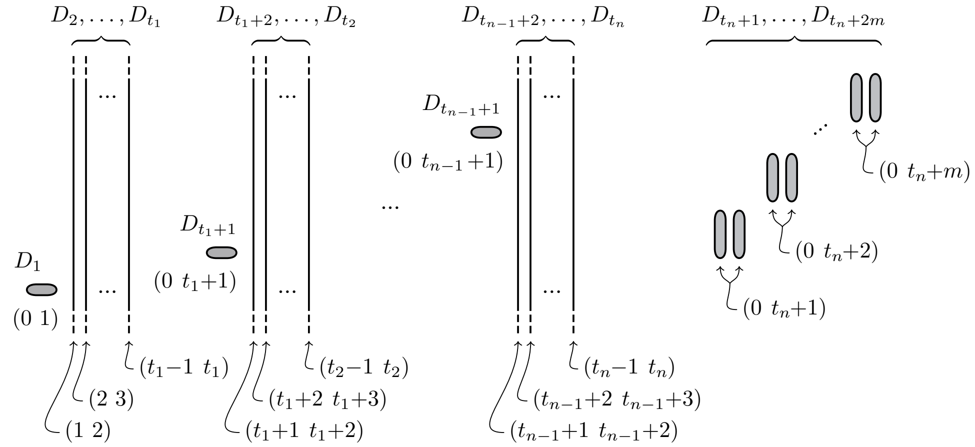


FIGURE 14.

A diagram of these branching disks with their monodromies is shown in Figure 14. Here, the vertical lines stand for flat vertical disks, transversal to the closed braid of Figure 13 in the upper part, where we cut open it, so that each string meets all them once, from right to left in the order. There is no such vertical disk for the 2-handles H^2_i such that K^2_i consists of only one string, that this $s_i = 1$ and $t_{i-1} + 1 = t_i$. Moreover, the disks representing D_{t_n+2j-1} and D_{t_n+2j} in the diagram are ε-displacements of the flat disk spanned by K^1_j in Figure 13, hence the K^2_i cross these three disks in the same way, for each $j = 1, \dots, m$. On the other hand, $D_{t_{i-1}+1}$ is a 2-disk expansion of a small horizontal arc $C_i \subset K^2_i$ placed at the beginning (on the left in Figure 13) of the lowermost string of K^2_i, for each $i = 1, \dots, n$.

The covering manifold M_1 can be thought as $B^4 \cup H^1_1 \cup \dots \cup H^1_m$. In fact, the disks D_{t_n+2j-1} and D_{t_n+2j} give raise to the 1-handle H^1_j formed by the sheet $t_n + j$, for each $j = 1, \dots, m$. All the other branching disks induce stabilization by addition of trivial sheets. An outline of M_1 (seen from the top) is drawn in Figure 15. We identify M_1 with $B^4 \cup H^1_1 \cup \dots \cup H^1_m$ in such a way that the blackboard framings relative to the Figures 13 and 15 coincide.

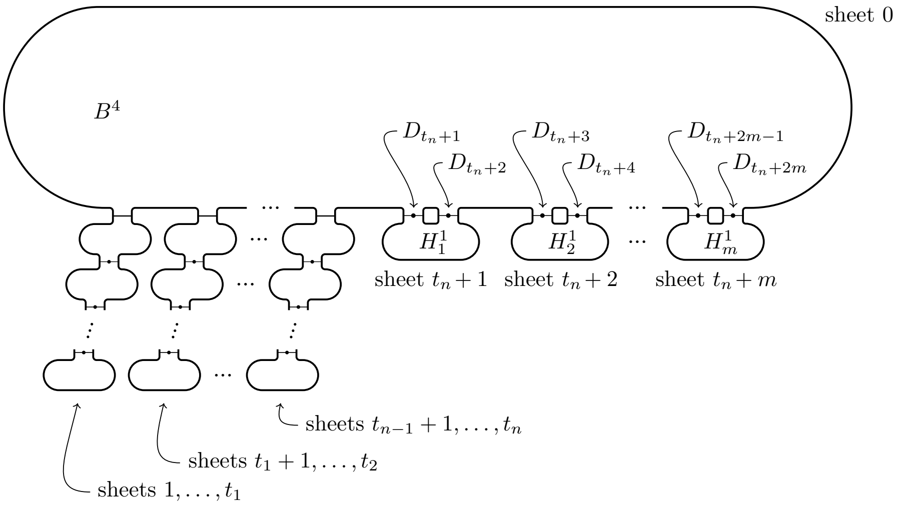


FIGURE 15.

Now, we modify the above branched covering $M_1 \to B^4$ to get the required simple branched covering $M \cong M_1 \cup H^2_1 \cup \ldots \cup H^2_n \to B^4$.

Following Montesinos [10] (see also [12]), we realize the addition of the 2-handles to M_1, by attaching an appropriate band B_i to the branching disk $D_{t_{i-1}+1}$, for each $i = 1,\ldots,n$. Namely, we define B_i as a ribbon band representing the blackboard framing along the arc $A_i = \mathrm{Cl}(K^2_i - C_i)$ in Figure 13.

This choice for the B_i works, since the following three properties are satisfied (cf. [10] or [12]): 1) the arc A_i meets the branching disks only at its endpoints, that belong to $\mathrm{Bd}\, D_{t_{i-1}+1}$; 2) the counterimage of A_i, with respect to the covering, is the disjoint union of $t_n + m - 1$ arcs and a simple loop $L_i = A'_i \cup A''_i \subset \mathrm{Bd}\, M_1$, where A'_i and A''_i are the liftings of A_i respectively starting in the sheets 0 and $t_{i-1} + 1$; 3) the link $L_1 \cup \ldots \cup L_n$ with the framings given by lifting the B_i's is equivalent (in $\mathrm{Bd}\, M_1$) to the link $K^2_1 \cup \ldots \cup K^2_n$ with the blackboard framings of Figure 13.

Actually, property 1 holds by construction, while property 2 can be easily verified by inspection, after observing that the product of the monodromies associated to the vertical lines of Figure 14 taken from right to left is the permutation $\pi = (1\ 2\ \ldots\ t_1)(t_1{+}1\ t_1{+}2\ \ldots\ t_2)\ldots(t_{n-1}{+}1\ t_{n-1}{+}2\ \ldots\ t_n)$.

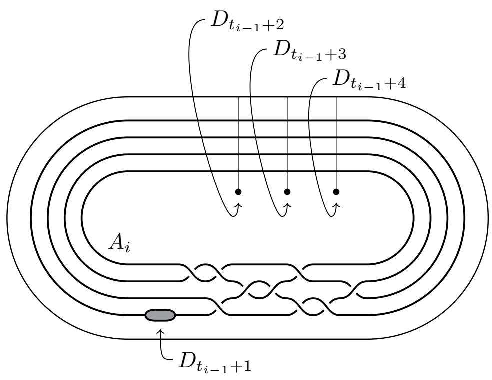


FIGURE 16.

So, we are left with proving property 3. For the moment, we focus on a single braid component K_i^2 disregarding the 1-handles. Figures 16 and 17 describe respectively the arc A_i and the loop L_i for a braid component K_i^2 with four strings, omitting the non-relevant branching disks and sheets.

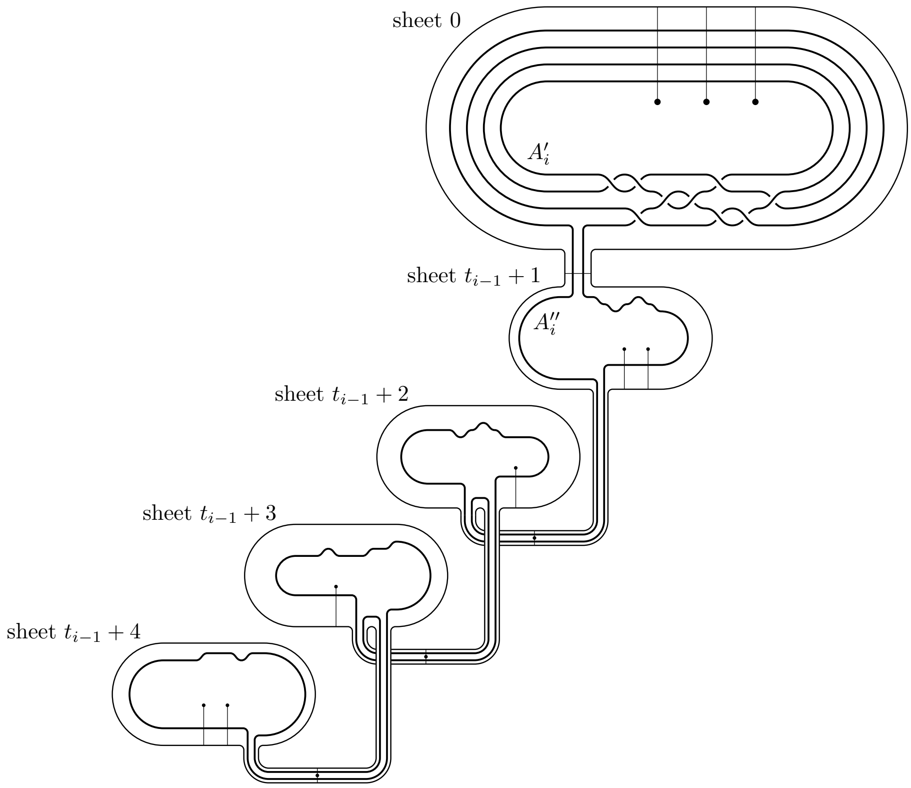


Figure 17.

By considering again the permutation π, we immediately see that A_i' is a copy of A_i entirely contained in the sheet 0 of the covering, while A_i'' is a trivial arc in the sheets $t_{i-1}+1, \ldots, t_i$, consisting of one string in each sheet (cf. Figure 17). Hence, L_i is isotopic to K_i^2.

Concerning the framing, we have that the blackboard framing along A_i in Figure 14 lifts to the blackboard framing along L_i in Figure 15 (cf. Figures 16 and 17), which in turn is equivalent to the blackboard framing of K_i^2 in Figure 13. The last equivalence is due to the fact that the isotopy between L_i and K_i^2 can be assumed to be regular with respect to the projection of figure 15.

At this point, we observe that all the K_i^2 can be considered simultaneously, since the L_i interact only in the sheet 0. Hence $L_1 \cup \ldots \cup L_n$ and $K_1^2 \cup \ldots \cup K_n^2$ are isotopic as blackboard framed links.

Finally, let us take into account the 1-handles. Figure 18 shows how the crossings of the A_i with the projections of $\operatorname{Int} D_{t_n+2j-1}$ and $\operatorname{Int} D_{t_n+2j}$ lifts to passages of the A_i' through the 1-handle H_j^1. In particular, no extra twist is added either in the link or in the framings. Then, the presence of the 1-handles does not affect our reasoning in any way, except for the fact that the arcs A_i' are no longer contained in the sheet 0 only, but they also traverse the sheets $t_n+1, \ldots, t_n+m$ forming the 1-handles.

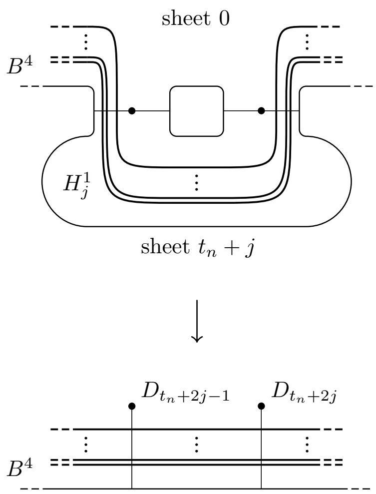


FIGURE 18.

A diagram of the resulting branching surface (cut open as above) is outlined in Figure 19. Here, the framed braid is the same of Figure 13, while the vertical disks are the ones of Figure 14, apart from different order, due to the sliding of the D_k's with $k \leq t_n$ from the upper part of the diagram (cf. Figure 16) to the lower one.

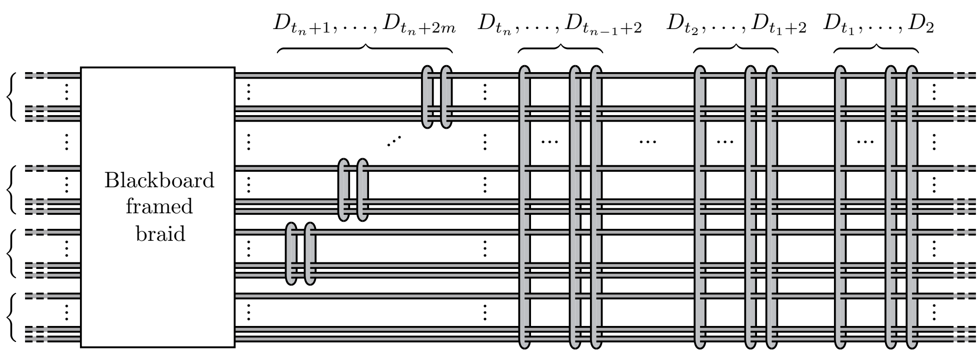


FIGURE 19.

3. Getting the universal surface

We begin this section, by explaining how the covering moves given in Section 1 can be used to symmetrize the branching surface of Figure 19.

Firstly, we modify any positive (resp. negative) crossing along the braid inside the box as described in the top (resp. bottom) part of Figure 20. In both cases, we perform eight Move 3 moves (cf. Figure 7) and then we isotope some of the resulting vertical disks. Then, we make all such crossings into ribbon intersections, by stabilization (followed by suitable isotopy) and crossing change, as shown in Figure 21 (of course, the covering degree d must be dynamically updated after each stabilization). We leave to the reader to check that the monodromies of the two bands forming each crossing are really distinct but not disjoint, like in Figure 21.

Secondly, we apply our Move 4 (cf. Figure 10) to the ribbon intersections we have just obtained, apart from the ones formed by the stabilizing disks. Moreover,

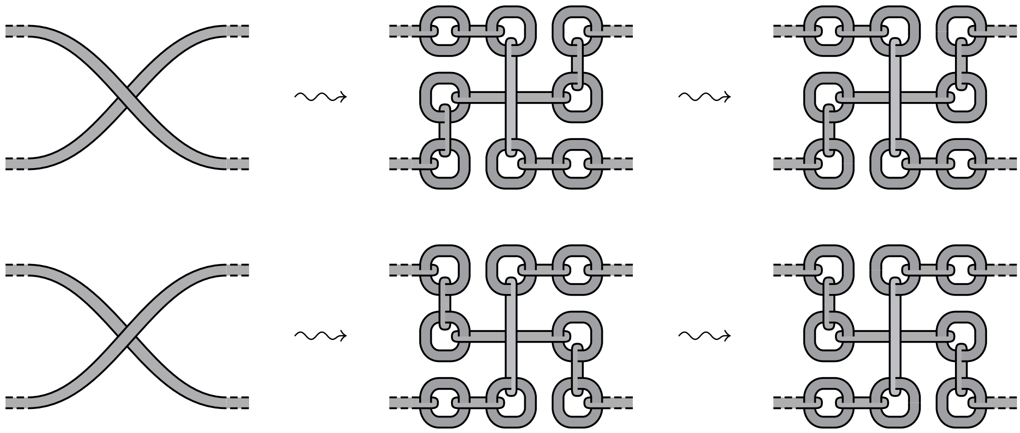


FIGURE 20.

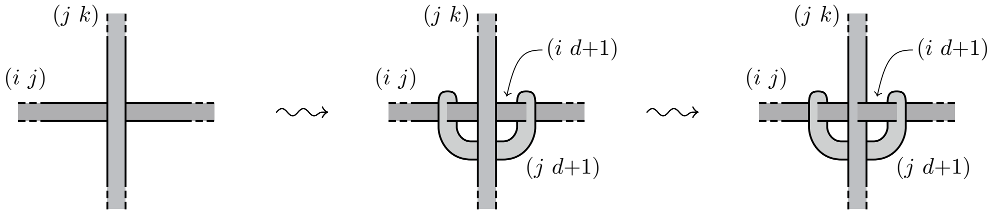


FIGURE 21.

we do the same on all the ribbon intersections which appear in Figure 19 outside the box. Also in this case, we leave to the reader to verify that the involved monodromies are distinct.

At this point, our diagram consists of: small annuli centered at some vertices of a rectangular grid; a certain number of horizontal and vertical bands running along some edges of the same grid; small stabilizing disks as in Figure 21 around some of the annuli. We emphasize that the bands do not form any ribbon intersection or crossing with each other.

Such a diagram can be easily completed to get the one depicted in Figure 22, where top and bottom ends are assumed to be trivially joined by bands passing in front of those already connecting left and right ends, and the pattern decorating the box has to be replicated at each potential crossing between the entering horizontal and vertical bands. Namely, it suffices to break the bands which take more that one

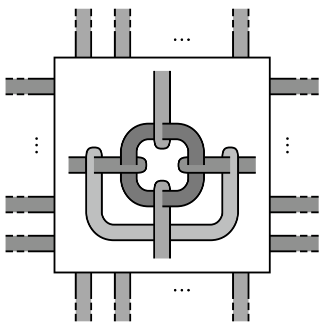


FIGURE 22.

grid edge, by using Move 3, and then to insert fake branching components (labelled with the identity) in the places that lack them. Of course, top and bottom grid lines have to be considered as if they were adjacent.

Now, thinking of B^4 as $B^2 \times B^2 \subset \mathbb{C}^2$, we place the diagram of Figure 22 in the torus $S^1 \times S^1$, in such a way that the rotations $r_1 : (z_1, z_2) \mapsto (e^{2\pi i/n_1} z_1, z_2)$ and $r_2 : (z_1, z_2) \mapsto (z_1, e^{2\pi i/n_2} z_2)$ permute respectively the rows and the columns of the $n_1 \times n_2$ pattern matrix inside the box (cf. [23] and [9]).

Then, we compose the branched covering represented by the diagram with the quotient by the action of r_2, to get a new branched covering $M \to B^4$, whose branching surface is given in Figure 23. Here, the rightmost disk is the branching surface of the quotient, while the box contains a $n_1 \times 1$ pattern matrix, which is the quotient of the one of Figure 22.

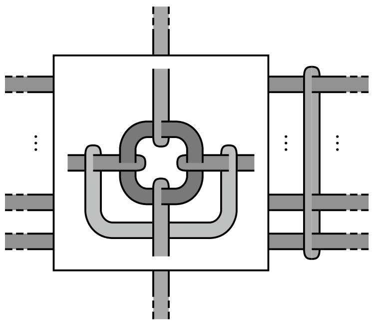


FIGURE 23.

Breaking the rightmost disk into n_1 disks, by Move 3 once again, and adding another fake branching annulus between top and bottom, we get the diagram of Figure 24, which can be still assumed to be r_1-invariant.

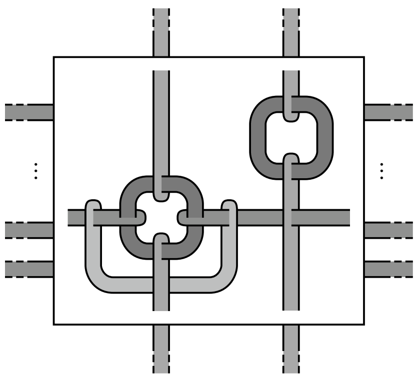


FIGURE 24.

Finally, we quotient by the action of r_1, in order to get the diagram of Figure 25, where the branching disk of this last quotient is the horizontal one. It is worth remarking that, by quotienting directly the diagram of Figure 23, one would get a singular point in the surface, due to the transversal intersection between the branching disks of the two quotients. This is the motivation for passing through the diagram of Figure 24.

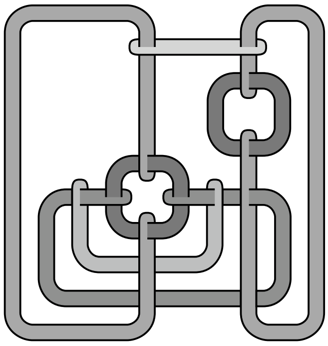


Figure 25.

Clearly, Figure 25 already represents a universal orientable ribbon surface. However, we conclude this section by simplifying such a universal surface a little. The intermediate steps of this simplification are described in Figure 26.

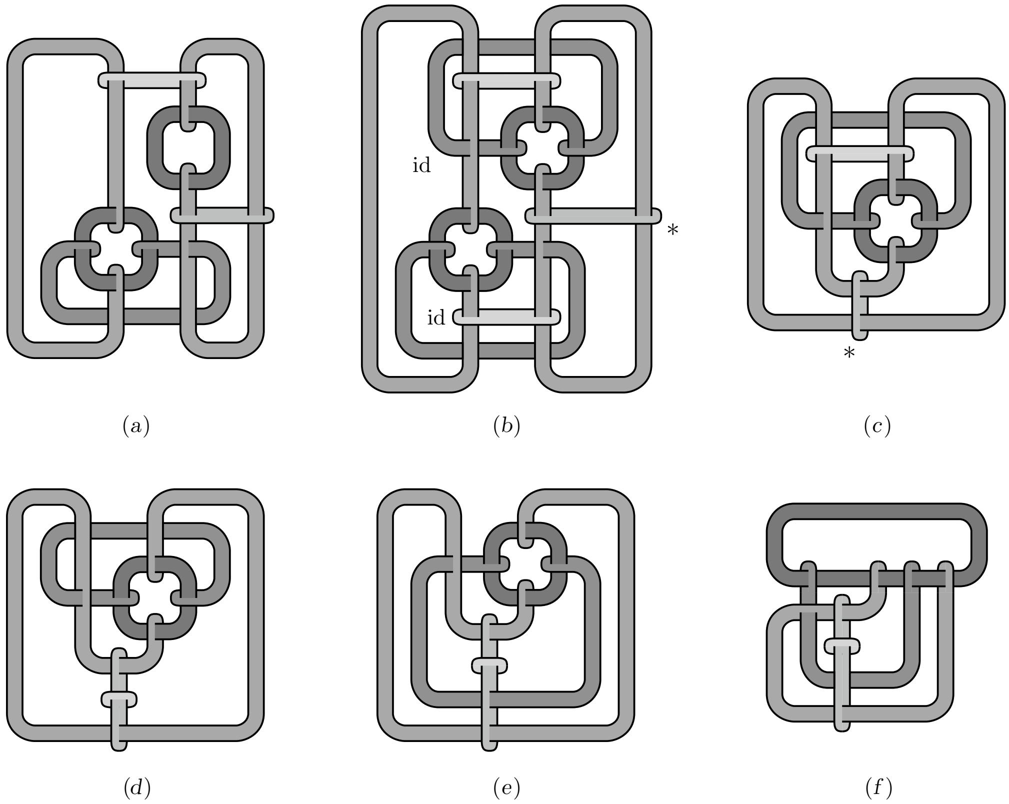


Figure 26.

We start with the surface in (a), which is isotopically equivalent to the one of Figure 25. Then, we add the two fake branching components labelled with the identity in (b), in order to make the surface symmetric with respect the center of the diagram (the disk marked with the asterisk can be thought as the fixed point set of the symmetry). In (c) we see the branching surface of the composition of our branched covering with that symmetry. Of course, this surface is still universal, but it has two components less than before. The surfaces in (d), (e) and (f) are all obtained by isotopy.

The simplified universal surface is depicted in Figure 27. To get it, we once again isotoped the last surface of Figure 26 just for pictorial reasons.

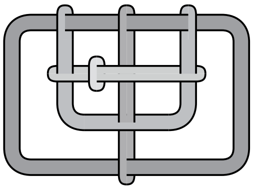


FIGURE 27.

4. Concluding remarks and questions

First of all, we observe that the universal surface of Figure 27 consists of one annulus and four disks, all trivially embedded in B^4. Moreover, disregarding the annulus, the four disks can be separated and isotoped in a symmetric position, so that they are cyclically permuted by a rotation of $\pi/2$ radians. Then, the quotient by the action of the rotation gives us a new universal ribbon surface with only three components, one annulus and two trivial disks. Unfortunately, the isotopy and the quotient force the annulus to wrap around the disks in a very unpleasant way and this makes resulting surface likely useless. Nevertheless, we know that the number of components can be reduced to three. However, the following question makes sense.

QUESTION 1. *Is there a 'reasonable' universal (possibly non-orientable) surface in B^4 with fewer that five components?*

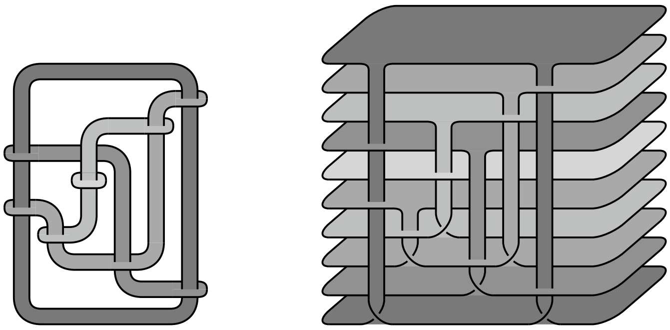
Even more, there is no reason to believe that three is the minimum number of components of a universal surface in B^4. In fact, it can easily be proved, by using signature, that there is no connected universal surface in S^4 (cf. [24] and [12]), but the same argument does not work in B^4. So, here is our second question.

QUESTION 2. *Does there exist any connected universal surface in B^4?*

On the other hand, at the cost of some more components, one could modify the construction carried out in Section 3, in order to get a different universal surface, such that branched coverings with all the branching indices equal to 2 would suffice for our representation theorem. The only branching indices bigger than 2 coming into that construction are indeed due to the rotations r_1 and r_2. Namely, the branching indices over the two disks fixed by such rotations (cf. Figures 23 and 25) are respectively n_1 and n_2. By the merging principle, each of these disks can be replaced by a pair of parallel disks labelled with suitable products of disjoint transpositions (the same argument used in [9] for the 3-dimensional case applies here). In this way, all the branching indices are reduced to 2.

Figure 28 (b) shows a braided version of our universal surface. It has been obtained by applying the Rudolph's braiding algorithm (cf. [21]) to the surface in the part (a) of the same figure, which is isotopic to the one of Figure 27.

Such way of seeing a universal surface is quite interesting, due to that fact that any covering $M \to B^4$ branched over a braided surface naturally induces a topological Lefschetz fibration $M \to B^2$ (cf. [14]). In particular, if the braided surface is



(a) (b)

FIGURE 28.

positive (quasi-positive in the Rudolph's terminology), that is all the bands between the sheets are positively twisted, then also the induced fibration is positive and M is a bounded Stein surface. One of the main ingredients in the proof of this fact is the Rudolph's theorem that positive braided surfaces are complex analytic (cf. [22]).

Since not all the 4-manifold considered here are Stein, no universal ribbon surface in B^4 can be isotopically equivalent to a positive braided surface. In Figure 28 (b) we see that only three of the six bands are positively twisted.

However, it has been proved in [14] that any compact Stein surface with boundary is a covering of B^4 branched over a positive braided surface. Then, the following question naturally arises.

QUESTION 3. *Does there exists a positive braided surface in B^4 which is universal for compact Stein surfaces with boundary?*

Finally, some trivial remarks about universal links. Obviously, the boundary of any universal surface in B^4 is a universal link in S^3. But, it is likely to be false that any universal link in S^3 is the boundary of a universal surface in B^4. Actually, it is not clear at all whether any universal link bounds some surface in B^4 allowing us to give a covering presentation of any closed orientable 3-manifold as the boundary of a 4-manifold. For example, we don't know what happens in the simplest case of the Borromean rings. So, we conclude with the following question.

QUESTION 4. *What universal links in S^3 bound a universal surface in B^4?*

References

[1] N. Apostolakis, *On 4-fold covering moves*, Algebraic & Geometric Topology **3** (2003), 117–145.

[2] I. Bernstein and A.L. Edmonds, *On the construction of branched coverings of low-dimensional manifolds*, Trans. Amer. Math. Soc. **247** (1979), 87–124.

[3] I. Bobtcheva and R. Piergallini, *Covering moves and Kirby calculus*, preprint 2004, math.GT/0407032.

[4] F. Harou, *Description en terme de revêtements simples de revêtements ramifiés de la sphère*, preprint 2002.

[5] H.M. Hilden, *Every closed orientable 3-manifold is a 3-fold branched covering space of S^3*, Bull. Amer. Math. Soc. **80** (1974), 1243–1244.

[6] H.M. Hilden, M.T. Lozano and J.M. Montesinos, *The Whitehead link, the Borromean rings and the knot 946 are universal*, Collect. Math. **34** (1983), 19–28.

[7] H.M. Hilden, M.T. Lozano and J.M. Montesinos, *Universal knots*, Lecture Notes in Math. **1144**, Springer-Verlag 1985, 25–59.

[8] H.M. Hilden, M.T. Lozano and J.M. Montesinos, *On knots that are universal*, Topology **24** (1985), 499–504.

[9] H.M. Hilden, M.T. Lozano, J.M. Montesinos and W.C. Whitten, *On universal groups and three-manifolds*, Invent. Math. **87** (1987), 441–456.

[10] H.M. Hilden and J.M. Montesinos, *Lifting surgeries to branched covering spaces*, Trans. Amer. Math. Soc. **259** (1980), 157–165.

[11] U. Hirsch, *Über offene Abbildungen auf die 3-Sphäre*, Math. Z. **140** (1974), 203–230.

[12] M. Iori and R. Piergallini, *4-manifolds as covers of S^4 branched over non-singular surfaces*, Geometry & Topology **6** (2002), 393–401.

[13] R. Kirby, *The topology of 4-manifolds*, Lecture Notes in Mathematics **1374**, Springer-Verlag 1989.

[14] A. Loi and R. Piergallini, *Compact Stein surfaces with boundary as branched covers of S^4*, Invent. math. **143** (2001), 325–348.

[15] J.M. Montesinos, *A representation of closed, orientable 3-manifolds as 3-fold branched coverings of S^3*, Bull. Amer. Math. Soc. **80** (1974), 845–846.

[16] J.M. Montesinos, *4-manifolds, 3-fold covering spaces and ribbons*, Trans. Amer. Math. Soc. **245** (1978), 453–467.

[17] J.M. Montesinos, *Representing 3-manifolds by a universal branching set*, Proc. Camb. Phil. Soc. **94** (1983), 109–123.

[18] J.M. Montesinos, *A note on moves and irregular coverings of S^4*, Contemp. Math. **44** (1985), 345–349.

[19] M. Mulazzani and R. Piergallini, *Lifting braids*, Rend. Ist. Mat. Univ. Trieste **XXXII** (2001), Suppl. 1, 193–219.

[20] R. Piergallini, *Four-manifolds as 4-fold branched covers of S^4*, Topology **34** (1995), 497-508.

[21] L. Rudolph, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helvetici **58** (1983), 1–37.

[22] L. Rudolph, *Algebraic functions and closed braids*, Topology **22** (1983), 191–202.

[23] W. Thurston, *Universal links*, preprint 1982.

[24] O.Ja. Viro, *Signature of branched covering*, Trans. Mat. Zametki **36** (1984), 549–557.