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Claspers and homotopy braids

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Introduction

Starting from the second half of the nineteenth century, knot theory has developed and spread up to become a useful tool in the most varied fields of applications, from physics to biology. Even if mathematical links arise from familiar objects and motivated by practical needs, several methods have been used in their study, leading to complex and deep mathematical topics such as, just to mention a few, the fundamental group of a link complement or the theory of braids.

The latter plays a fundamental role in the study of links: braids are not only a nice way to visualise links but they also provide an algebraic structure that allows to study in detail their intrinsic properties. Moreover, the study of braids has led to some polynomial invariants of links (for example, Jones polynomial) but still not to their complete classification.

In this work we will focus on the notion of *link-homotopy*, introduced by Milnor in 1954 in the context of knot theory: it is an equivalence relation on links that allows continuous deformations during which each component can self-intersect, while different components must remain disjoint at all times. It is clear that any knot (that is, a one-component link) is link-homotopic to the trivial one, but for links with more than one components this relation turns out to be quite complicated. In particular, we are interested here in the study of braids up to link-homotopy.

The *homotopy braids group* was introduced by Goldsmith in [5], where she gives an example of a non-trivial braid up to isotopy, that is trivial up to link-homotopy and a presentation of this group. This structure is also studied by Habegger and Lin [7]. Our purpose is, following [6], to reinterpret the homotopy braids group in terms of *claspers*: developed by Habiro in [8], these are surfaces with additional structures on which surgery operations can be performed. In [8], Habiro describes a set of geometric operations on claspers that yield equivalent surgery results, i.e. *clasper calculus* up to isotopy. His work can be refined (see for example [13]) and used for the study of knotted objects up to link-homotopy: this *homotopic clasper calculus* will be a key tool for our aims.

The rest of this thesis is organized as follows. In Chapter 1 we recall some basic definitions and important results about links and surgery along a link in a 3-manifold, following [19]. Then we quickly review braids up to both isotopy and link-homotopy and the correspondent algebraic structures (Sections 1.2-1.3). In order to emphasize the importance of braids in the study of links, we report here also Alexander and Markov's theorems.

Chapter 2 is dedicated to a brief presentation of clasper theory (which can be found in [8]) and clasper calculus up to link homotopy: a fundamental lemma from [4], combined with Habiro's work, gives us a set of geometric operations on claspers having link-homotopic surgery results.

Chapter 3 focuses on the reinterpretation of braids in terms of claspers; in Section 3.1 we define *comb-claspers* and, after an algebraic interlude in Section 3.2 we conclude with a linear, faithful representation of the homotopy braid group, whose injectivity is proved using homotopic clasper calculus.

Chapter 1

Preliminaries

We start with some preliminary tools that we will need in order to define the so-called *claspers*.

Definition 1.1. We call an *n-component link* $L \subset \mathbb{R}^3$ the image via a smooth embedding $f : \mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1 \hookrightarrow \mathbb{R}^3$ of n disjoint closed curves. A link with only one component is usually called a *knot*. An *oriented link* is a link endowed with a fixed orientation.

Recall that given two differentiable maps $g, g' : \mathbb{R}^3 \to \mathbb{R}^3$, an isotopy between g and g' is a continuous map $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that, if we set $h_t(x) = H(x,t)$ for any $t \in [0,1]$, we have that $h_0 = g$, $h_1 = g'$ and h_t is a homeomorphism for all $t \in [0,1]$. If H is an isotopy between g and g' we denote it by $H : g \cong g'$. A well-known equivalence relation between links is given by the following.

Definition 1.2. We say that two links L and L' are *equivalent* if they are related by an ambient isotopy, that is, there exists a diffeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L) = L'$ and an isotopy $H : \mathrm{id}_{\mathbb{R}^3} \cong h$.

Given a link $L \subset \mathbb{R}^3$, up to ambient isotopy we can always assume that its projection $\pi_{|_L} : L \to \mathbb{R}^2$ is regular and injective, except for a finite number of transverse double points.

Then, the *diagram* D of a link L is given by its planar projection, in which at each crossing we distinguish the over-strand from the under-strand as in Figure 1.2. Note also that any diagram represents a link uniquely determined up to isotopy (for more details, see [19, Paragraph 3.E]).

Definition 1.3. Let D, D' be two link diagrams of L, L' respectively. We say that D and D' are *equivalent* if they coincide up to a finite sequence of planar isotopies (that preserve crossings informations) and Reidemeister moves, which locally modify the structure of the diagram as depicted in Figure 1.1.

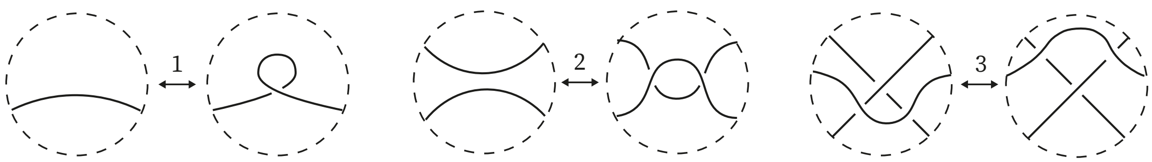


Figure 1.1: Reidemeister moves for link diagrams

Then we have:

Theorem 1.4. *(Reidemeister's Theorem) Two links are equivalent if and only if their diagrams are equivalent.*

Now let $L \subset \mathbb{R}^3$ be an n-component link, $L = \mathrm{Im}(f)$, where $f : \mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1 \hookrightarrow \mathbb{R}^3$ is smooth. A tubular neighbourhood $N(L)$ of L is the image via a smooth embedding of n disjoint solid tori:

$$\iota : (D^2 \times \mathbb{S}^1) \sqcup \cdots \sqcup (D^2 \times \mathbb{S}^1) \hookrightarrow \mathbb{R}^3$$

such that $L = \mathrm{Im}(f) = \iota(\{0\} \times \mathbb{S}^1 \sqcup \cdots \sqcup \{0\} \times \mathbb{S}^1)$. Remember also that the boundary of the solid torus is $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

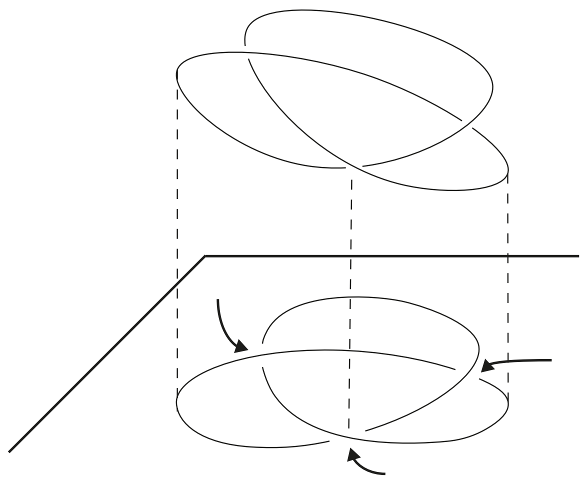


Figure 1.2: Planar diagram of the trefoil knot. Notice that crossings correspond to double points of the projection

Another important equivalence relation between links, but weaker than isotopy, is given by the notion of *link-homotopy*, introduced by Milnor in [15].

Definition 1.5. Two links are link-homotopic if they can be transformed into each other by a sequence of self-crossing changes and ambient isotopies. More particularly, crossing changes between the same components of the same link are allowed, while different components remain disjoint throughout the transformation.

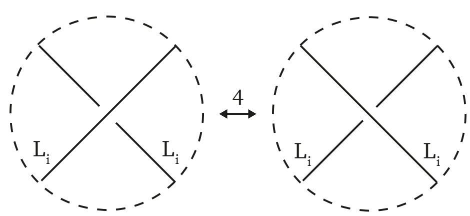


Figure 1.3: A homotopic Reidemeister move.

From link-homotopy point of view, we recover an analogue of Reidemeister's Theorem, but we need to add a further move of link diagrams, represented in Figure 1.3, which shows a crossing change between the i-th component of an n-link $L = L_1 \cup \cdots \cup L_n$ and itself. We call moves 1-2-3 of Figure 1.1, together with move 4 of Figure 1.3, *homotopic Reidemeister moves.*

Theorem 1.6. *(Homotopic Reidemeister Theorem) Two links L, L' are link-homotopic if and only if their diagrams are related by a finite sequence of planar isotopies (preserving crossings informations) and homotopic Reidemeister moves.*

1.1 3-manifolds and surgery on links

We now introduce a construction used to modify 3-manifolds called *Dehn surgery*: the process takes as input a 3-manifold together with a link. In what follows, if not specified, we will denote with M a smooth, compact, connected and oriented 3-manifold.

Definition 1.7 (*Drilling*)**.** Let M be a 3-manifold and $L = L_1 \cup \cdots \cup L_n \subset M$ an n-component link. Consider an open tubular neighbourhood $N(L)$ of L in M and remove it from M. Then the 3-manifold $M \setminus N(L)$ is the result of *drilling M along L*.

Notice that $\partial(M \setminus N(L))$ is given by $T_1 \cup \cdots \cup T_n$, where each T_i is a 2-torus. Before moving on, we may give the following.

Definition 1.8 (*Gluing*)**.** Let X and Y be topological spaces, $Z \subset X$ a subspace and $f : Z \to Y$ a continuous map. Consider $X \sqcup Y$ and the equivalence relation generated by $z \sim f(z)$ for any $z \in Z$. The space

$$X \cup_f Y \coloneqq {}^{(X \cup Y)}\big/_{\sim}$$

with the quotient topology is said to be obtained by *gluing X and Y along f*.

Definition 1.9 (*Filling*)**.** Let $M \setminus N(L)$ be a 3-manifold drilled along some n-link $L \subset M$. Consider a homeomorphism $h_i : T^2 \to T_i$. Then, we may glue one solid torus and T_i along

h_i. This process is called *Dehn filling*.

Definition 1.10. A *Dehn surgery* on a 3-manifold M along a link $L \subset M$ consists on drilling M along L together with filling on all the components of the boundary corresponding to the link.

In particular, if we specify:

1. n disjoint closed tubular neighbourhoods N_i of each component L_i of L in $\operatorname{int} M$;

2. a simple closed curve γ_i in each ∂N_i,

we may construct the 3-manifold

$$M' = (M \setminus (\operatorname{int} N_1 \cup \cdots \cup \operatorname{int} N_n)) \cup_h (N_1 \cup \cdots \cup N_n),$$

where $h = \bigsqcup_i h_i$ is a union of homeomorphisms $h_i : \partial N_i \to \partial N_i \subset M$, each of which takes a meridian curve μ_i of N_i onto the specified γ_i. In this situation, the 3-manifold M' is said to be the result of a Dehn surgery on M along the link L with *surgery instructions* (1) and (2).

The following Proposition ([19, Theorem 2.C.16]) represents an important result about knots in the torus and adds an essential detail in describing surgery on a manifold. Recall that if X is a topological space, $H_1(X)$ denotes the first homology group of X.

Proposition 1.11. *Two knots $K, K' \subset T^2$ are ambient isotopic in T^2 if and only if $[K] = \pm [K']$, where $[\omega]$ denotes the class of ω in $H_1(T^2)$.*

Proof. The condition is necessary since an ambient isotopy between K and K' restricts to a homotopy taking K onto K'.
Conversely, Recall that $H_1\left(T^2\right) = \mathbb{Z} \oplus \mathbb{Z}$. If $[K] = \pm [K'] \neq (0, 0)$, let $h : T^2 \to T^2$

such that $h_*([K]) = h_*([K']) = (0, \pm 1)$. Then (see [19, Theorem 2.C.8]) there exists an ambient isotopy $(g_t : T^2 \to T^2)_{t \in [0,1]}$ such that $g_1(h(K)) = h(K')$. Then $(h^{-1} g_t h)_{t \in [0,1]}$ is an ambient isotopy of T^2 with $h^{-1} g_1 h(K) = K'$ as required. □

By the above proposition, the resulting manifold M' of a Dehn surgery is uniquely determined if we specify the homotopy class of each γ_i: indeed, two distinct simple and closed homotopic curves $\gamma_i, \gamma_i' \subset T^2$ are ambient isotopic in T^2 and hence yields isotopic homeomorphisms $h_i, h_i' : \partial N_i \to \partial N_i$.

Example 1.12 (Dehn's construction of a homology sphere)**.** Let N be a tubular neighbourhood of a right-handed trefoil K and let γ be the curve on N as depicted in figure 1.4. Consider a homeomorphism

$$h : \partial(\mathbb{S}^1 \times D^2) \to \partial N$$

which takes a meridian $\{*\} \times \mathbb{S}^1$ onto γ and form

$$Q^3 = \left(\mathbb{S}^3 \setminus \operatorname{int} N\right) \cup_h (\mathbb{S}^1 \times D^2)$$

sewing a solid torus to the knot exterior via h.

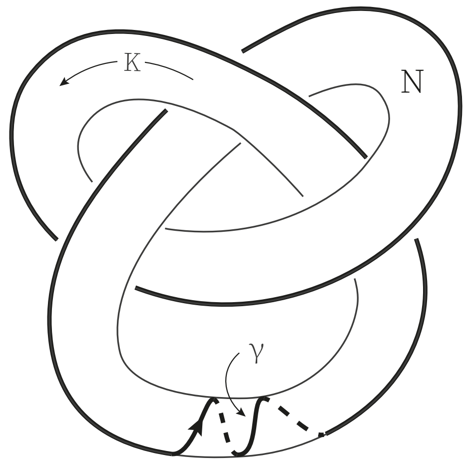


Figure 1.4: Example 1.12

Then, Q^3 is a connected, closed, orientable 3-manifold. Its homology groups can be calculated using a Mayer-Vietoris sequence and will be found to coincide with those of $\mathbb{S}^3$.

Let us proceed with some further details about surgery instructions in $\mathbb{S}^3$. Consider a knot $K \subset \mathbb{S}^3$ and let $N = N(K)$ be a tubular neighbourhood of K (a solid torus). Let $X = \overline{\mathbb{S}^3 \setminus N}$. Then, ([19, 2.E.7]) up to ambient isotopy of N, there is a unique longitude which is homologically trivial in X. Moreover, there is a homeomorphism

$$h : D^2 \times \mathbb{S}^1 \to N,$$

also unique up to ambient isotopy, such that $h(\{*\} \times \mathbb{S}^1)$ is homologically trivial in X. Hence, each component L_i of an oriented link L has a preferred reference frame for a tubular neighbourhood N_i, in which the longitude λ_i ($[\lambda_i] = (1,0) \in H_1(\partial N_i)$) is oriented as L_i and the meridian μ_i ($[\mu_i] = (0,1) \in H_1(\partial N_i)$) has linking number $+1$ with L_i. Any prescribed curve $\gamma_i \subset \partial N_i$ can therefore be written as

$$h_* ([\mu_i]) = \pm [\gamma_i] = a_i \lambda_i + b_i \mu_i,$$

depending on how γ_i is oriented. Notice that $b_i = \ell(L_i, \gamma_i)$, where $\ell(L_i, \gamma_i)$ is the linking number between L_i and γ_i. Take the ratio

$$r_i = \frac{b_i}{a_i}$$

and call it the *surgery coefficient* associated to the component L_i. Since γ_i is a simple closed curve for every i, we need a_i and b_i to be coprime integers. If $a_i = 0$ then $b_i = \pm 1$ and we set $r_i = \infty$. Moreover, when the ratios r_i are all integers we call the surgery an *integral surgery.*

Remark 1.13. *Since reversing orientation of the ambient space $\mathbb{S}^3$ changes all linking numbers signs, we will always work with a fixed orientation.*

Therefore, any link L in $\mathbb{S}^3$ with rational numbers attached to its components completely determines a surgery which yields a closed oriented 3-manifold.

Definition 1.14. A choice of such a fraction for each component of a link L is called a *framing* of L. A link L with a fixed framing is called a *framed link.*

Lickorish (1962) and (independently) Wallace (1960), proved the following

Theorem 1.15. *Every closed, orientable, connected 3-manifold may be obtained by surgery on a link in $\mathbb{S}^3$. Moreover, one may always find such a surgery presentation in which the surgery coefficients are all $+1$ and the individual components of the link are unknotted.*

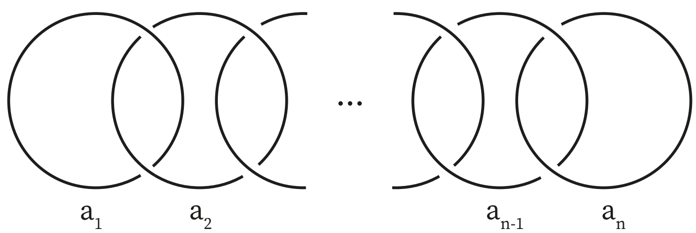
In practice, we describe a manifold by drawing a diagram of a link L and writing the surgery coefficients near their respective components.

Example 1.16. Let L be the trivial knot in $M = \mathbb{S}^3$ and denote by M' the result of the

Dehn surgery on M along L. Then the surgery coefficient $r = b/a$ determines:

$$\begin{aligned} M' &\cong \mathbb{S}^2 \times \mathbb{S}^1 && \text{if} \quad r = 0, \\ M' &\cong \mathbb{S}^3 && \text{if} \quad r = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \dots, \infty. \end{aligned}$$

Moreover, if L is the trivial n-link and $a_1, a_2, \dots, a_n$ are integers, then the surgery description



yields $M' = L(p, q)$, where p, q are coprime integers determined by the continued fraction decomposition

$$\frac{p}{q} = a_n - \cfrac{1}{a_{n-1} - \cfrac{1}{a_{n-2} - \cdots \cfrac{1}{a_2 - \cfrac{1}{a_1}}}}$$

and $L(p, q)$ is the *Lens space of type* (p, q).

In [19, Paragraph 9.H] is described how to modify a surgery presentation of a manifold. In particular, we can always assume that the surgery coefficients are all integers. A standard way to represent an integral surgery without explicitly indicate the coefficients is to use the so-called *blackboard framing*. The blackboard framing is obtained by converting each link component to a ribbon, namely adding a parallel strand to each component, so we get something like the image via a smooth embedding of n disjoint annuli in $\mathbb{R}^3$ (provided the ribbon does not form a Möbius band). Each annulus is a component of the framed link. Then, as for links, given a regular projection we can indicate a crossing structure using line breaks to obtain a diagram of the framed link. In particular, we can assume that every crossing in a diagram is one of the following types:

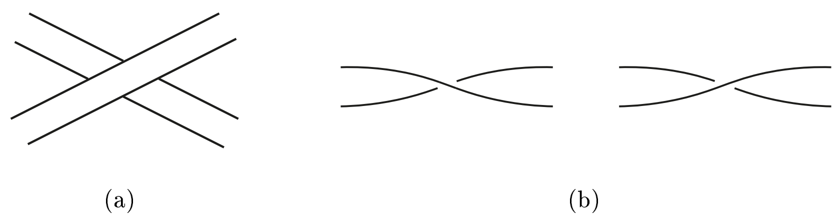


Figure 1.5: Possible crossings in framed links diagrams.

A type (a) crossing involves two different segments of the framed link (eventually segments of the same component); a type (b) crossing involves a single segment and records some "twisting" of the framed link. Observe that the twists can be eliminated in pairs by isotoping the framed link as represented in the following figure:

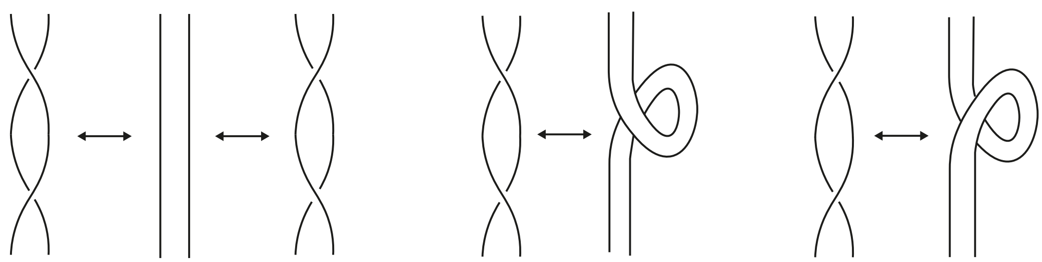


Figure 1.6: Simplifying twists of framed links.

The result of this observation is that we can assume that our drawing consists of a "thickening" of a link diagram as indicated in Figure 1.7.

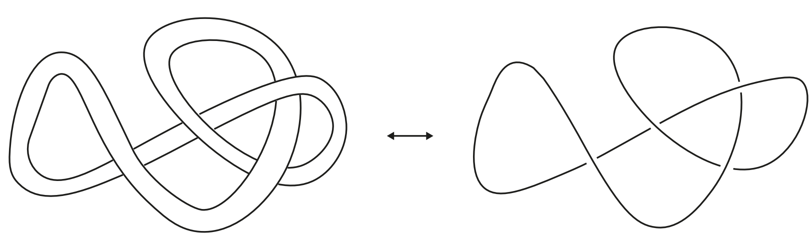


Figure 1.7: An example of a "thickening" of a link diagram

Thus, every framed link can be represented by a link diagram. Conversely, every link diagram gives rise to a framed link by thickening the drawing.

Once we established what constitutes a positive and/or negative crossing, the framing is given by the linking number of the two edges of the ribbon, considered as a two-components link. In other words, the surgery-coefficients become the number of (signed) twists of the added strand around the original one.

As was the case with knots and links, we need to understand how diagrams that represent isotopic framed links are related to each other, that is, we need an analogue of the Reidemeister moves and Reidemeister's Theorem for framed links. A type 1 Reidemeister move clearly changes the blackboard framing, as it is shown in Figure 1.8, but the other two moves do not.

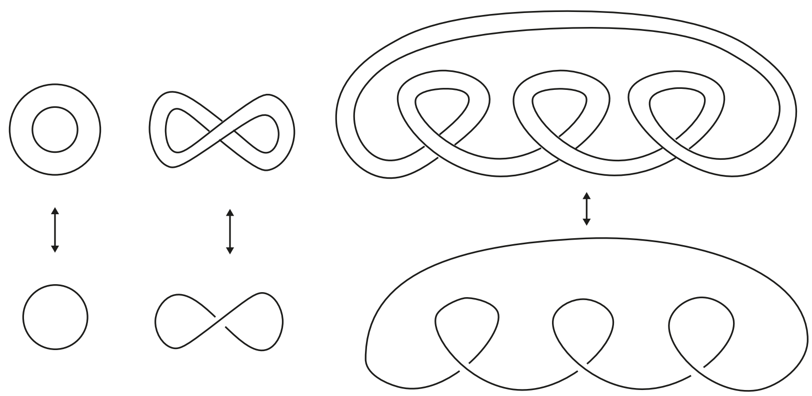


Figure 1.8: Examples of inequivalent framed links with their corresponding diagram.

Hence, if we modify the first Reidemeister move as depicted in Figure 1.9 (see for instance [12]), we get a result for links diagrams with blackboard framing similar to the Reidemeister theorem.

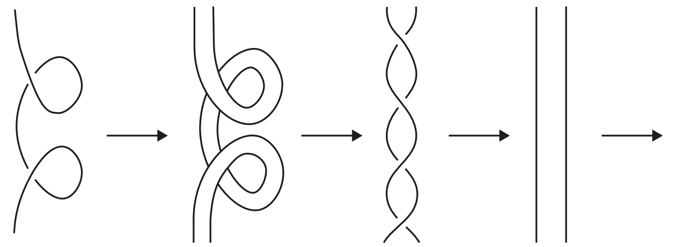


Figure 1.9: Local representation of the first Reidemeister move for framed links

Theorem 1.17. *Two framed links are equivalent (i.e. ambient isotopic) if and only if their diagrams are related by a finite sequence of framed Reidemeister moves. That is*

$$\frac{\{\textit{framed links}\}}{\textit{isotopy}} \cong \frac{\{\textit{link diagrams}\}}{\textit{framed Reidemeister moves}}$$

Proof. See [12, Theorem 3.5]. $\square$

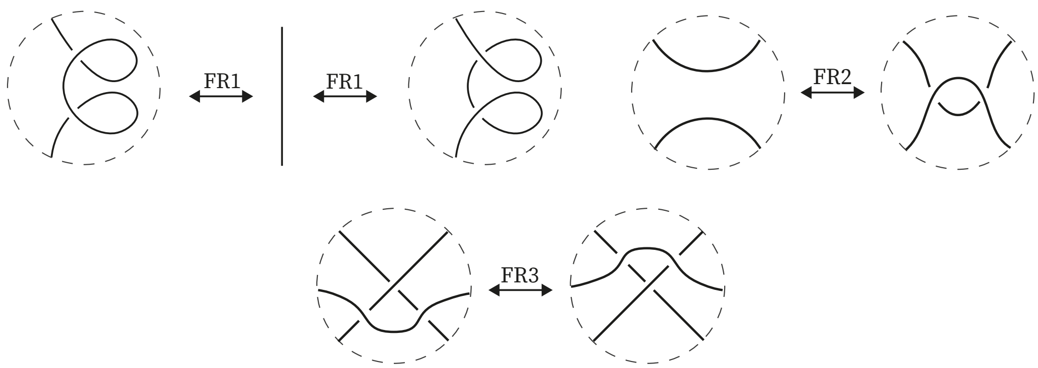


Figure 1.10: Framed Reidemeister moves

Finally notice that for integral surgery, since the linking number is an isotopic invariant of links, two equivalent framed links have the same framing. Hence, surgeries along equivalent framed links yield diffeomorphic resulting manifolds.

1.2 Braids group

Even if much progress has been achieved in the classification of links, there are still many aspects highlighting the difficulty of this problem. Braids are a closely related and visually similar objects, but simpler enough to have a complete classification and a high-detailed profile of their properties. In this sections we'll summarise the most important (and needed for our purposes) results in studying braids up to isotopy and homotopy.

Firstly, a couple of algebraic tools. For any a, b in a group, let $[a, b] \coloneqq aba^{-1}b^{-1}$. We will denote by $F(x_1, \ldots, x_n)$, or simply F_n, the *free group* of rank n with generators $x_1, \ldots, x_n$ and by RF_n the *reduced free group*, namely the quotient

$$RF_n \coloneqq {}^{F_n}\!/_{J_{F_n}}, \text{ with } J_{F_n} \coloneqq \langle\!\langle \{[x_i, \lambda x_i \lambda^{-1}] \mid i \in \{1, \ldots, n\},\ \lambda \in F_n\} \rangle\!\rangle$$

where $\langle\!\langle \{y_1, \ldots, y_n\} \rangle\!\rangle$ stands for the subgroup *normally generated* by $\{y_1, \ldots, y_n\}$.

Let now $D \subset \mathbb{R}^2$ be the unit disk centered at $(\frac{1}{2}, 0)$ and fix n equally spaced points $\{p_i\}_{i \leq n} \subset D$ lying on the x-axis. Let also I be the unit interval $[0, 1]$ and denote by $I_1, \ldots, I_n$ n copies of I.

Definition 1.18. An *n-component braid* β is the image of a smooth proper embedding

$$(\beta_1, \ldots, \beta_n) : \bigsqcup_{i \leq n} I_i \to D \times I$$

such that $\beta_i(0) = (p_i, 1)$ and $\beta_i(1) = (p_{\pi_\beta(i)}, 0)$ where π_β is the permutation of $\{1, \ldots, n\}$ associated to β. We also require the embedding to be monotonic, which means that $\beta_i(1 - t) \in D \times \{t\}$ for any $i = 1, \ldots, n$ and $t \in [0, 1]$. If the permutation associated to β is the identity, we call β a *pure braid*.

As usual, braids will be represented here through their projection on $I \times \{0\} \times I$, in which we specify informations about under-crossings and over-crossings.

Definition 1.19. We say that two braids are *isotopic*, and we write $\beta \cong \beta'$, if they can be deformed one to another by an ambient isotopy $H = (h_t)_{t\in[0,1]}$ of $D \times I$ relative to $\partial(D \times I)$ that is,

$$h_{t|_{\partial(D\times I)}} = \mathrm{id}_{\partial(D\times I)}$$

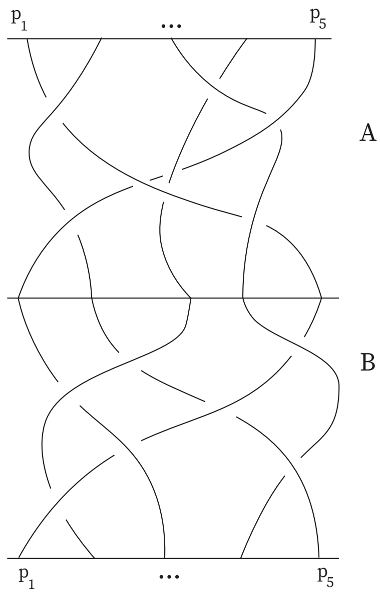
for any $t \in [0, 1]$.

Observation. We can assume that the isotopy of Definition 1.19 is level-preserving. Moreover, the condition for two braids β, β' to be isotopic is equivalent to say that there exists a continuous family $(B_t)_{t\in[0,1]}$ of braids such that $B_0 = \beta$ and $B_1 = \beta'$. In other words, there exists a continuous map

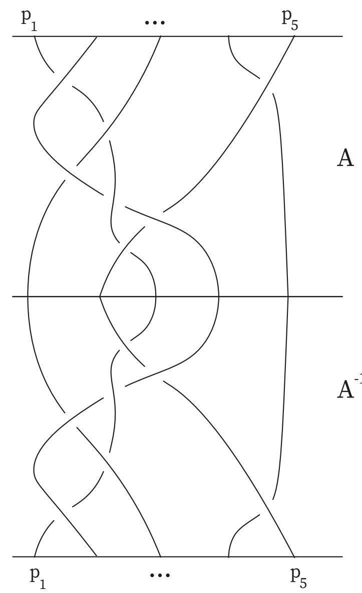
$$J : B \times I \to D \times I$$

such that $J(B \cap D_s, t) = B_t \cap D_s \ \forall t, s \in [0, 1]$, where $D_s := D \times \{s\}$.

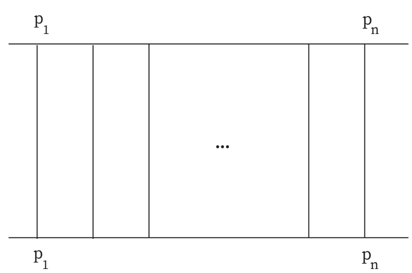
Notice that isotopy induces equivalence relation on the sets of n-braids. The choice of a set of initial and final endpoints allows us to compose two braids, basically attaching the initial endpoints of the second braid to the terminal endpoints of the first one and fitting the result back in the unit cylinder (Figure 1.11(a)). With respect to this composition, the trivial pure n-braid (Figure 1.11(c)) represents the identity element 1_n. Moreover, if we consider a braid A and its reflection with respect to the horizontal plane (Figure 1.11(b)), denoted by A^{-1}, we have that the composition AA^{-1} can be untangled and hence $AA^{-1} = 1_n$. The associativity property is easy to check. Observe also that the isotopy relation respects braids composition; namely, if we take $\beta_1, \beta_2, \beta_1', \beta_2' \in B_n$, where $\beta_i' \cong \beta_i$ for $i = 1, 2$, then $\beta_1'\beta_2' \cong \beta_1\beta_2$, so the composition operation is well defined on the set of isotopy braids. Thus, the set of isotopy classes of n-braids endowed with the operation described above is a group, which E. Artin defines in [1] and denotes by B_n.



(a) Braids product



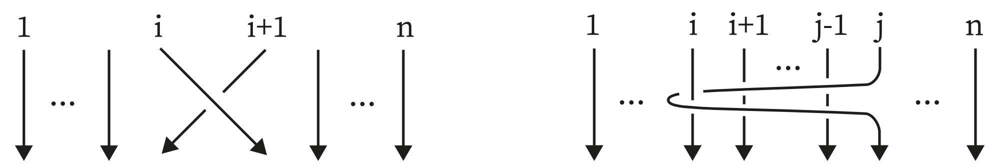
(b) Braids inverse element



(c) The trivial braid 1_n

Figure 1.11

Given an n-braid β, in its projection two crossings may occur at the same height. With a slight deformation of braid (an isotopy) we can obtain a diagram in which this does not happen, so that all crossings are arranged at different heights. If we "cut" the diagram into sufficiently small horizontal sections in such a way each section contains only one crossing, the original braid is obtained from all those sections by tying them together again. This procedure allows us to write every braid in B_n as a product of the $n-1$ generators $\sigma_1, \ldots, \sigma_n$ and their inverse, where the braids σ_i consist of passing the i-th strand once over the $(i+1)$-st one as illustrated in Figure 1.12.



(a) The braid generator σ_i.

(b) The pure braid generator A_{ij}

Figure 1.12

It is well known that in terms of these generators we obtain a presentation of the braid group B_n:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_n \;\middle|\; \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |j-i| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \ldots, n-2. \end{array} \right\rangle .$$

Braids are deeply related to the study of links. In particular, we can associate an oriented link to every braid, as explained in the next definition.

Definition 1.20. Let $\beta \subset D \times I$ be a braid and define the map

$$\begin{aligned} \eta : D \times I &\to \mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R} \\ (x, y, z) &\mapsto \eta(x, y, z) := (x\cos(2z-1)\pi, x\sin(2z-1)\pi, y) \,. \end{aligned}$$

Then $\hat{\beta} := \eta(\beta) \subset \mathbb{R}^3$ is an oriented link that we call the *closure* of the braid β.

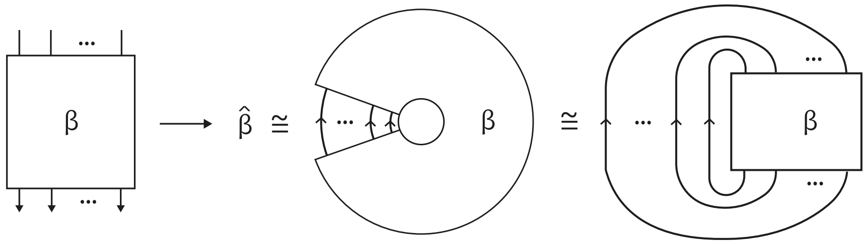


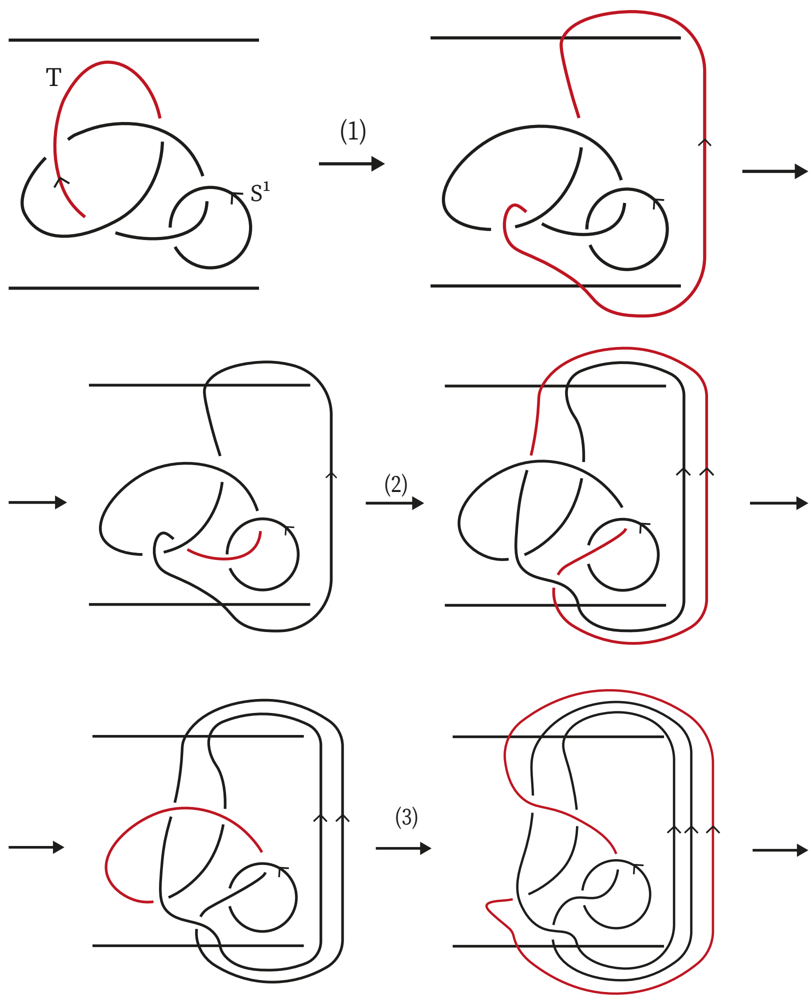
Figure 1.13: A graphic representation of the closure of a braid

The closure of an n-braid β can be also seen as an oriented link obtained by joining each point $(p_i, 0)$ with the corresponding endpoint $(p_i, 1)$ through n arcs contained in the xz plane, disjoint from each other and from the interior of $D \times I$. Figure 1.13 intuitively shows this equivalence based on link isotopy. Moreover, it can be shown that isotopic braids give link-isotopic braids closures. Two fundamental results about this topic are due respectively to Alexander and Markov

Theorem 1.21. *(Alexander) For any link $L \subset \mathbb{R}^3$ there exist $n \in \mathbb{N}$ and a braid $\beta \in B_n$ such that $L \cong \hat{\beta}$.*

Proof. For a detailed proof, see [3, Theorem 6.5]. □

Example 1.22. Let T be the trefoil knot and consider the oriented link $L = T \sqcup \mathbb{S}^1$. Figure 1.14 show the procedure to write L as the closure of a braid. In particular we have $L \cong \hat{\beta}$, where $\beta = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1}\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3$. In the last picture we have re-drawn β to make the crossings easier to recognise.


T
S¹
(1)
(2)
(3)

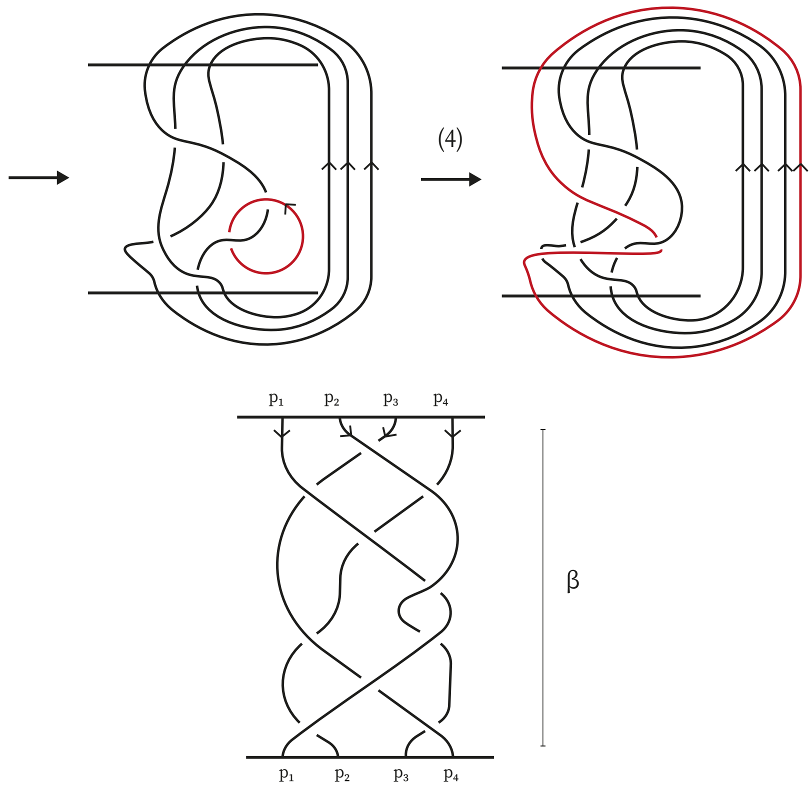


Figure 1.14: Turning $T \sqcup \mathbb{S}^1$ into the closure of a braid.

Markov's theorem gives us a criterion to establish whether the closures of two braids are isotopic.

Definition 1.23. A *Type 1 Markov move* on an n-braid β, also called braids *conjugation*, replaces β by a conjugate $\gamma\beta\gamma^{-1}$, where γ is any n-braid.

Definition 1.24. A *Type 2 Markov move* on an n-braid β, also known as braids *stabilisation*, replaces β by the $(n+1)$-braid $\beta\sigma_n$ or $\beta\sigma_n^{-1}$.

Theorem 1.25. *(Markov) Let $\beta \in B_n$ and $\beta' \in B_m$. Then $\hat{\beta}$ and $\hat{\beta}'$ are equivalent as oriented links if and only if there exists a finite sequence of Markov moves on braids that transforms β into β'.*

Proof. See [3]. □

Combining these two theorems we get a powerful tool to check if two links L, L' are equivalent: just find two braids β, β' such that $\hat{\beta} \cong L$, $\hat{\beta}' \cong L'$ and check if they are related by a sequence of Markov moves. However, a slight problem is due to the fact that, at present, there is no known algorithm to determine if two braids can be transformed into each other using Markov moves.

Let us now focus on the *pure n-braid group* $P_n \subset B_n$. From [2] we know that every pure braid is isotopic to one in *normal form*. To describe this normal form, we first define the generators $A_{i,j}$ of P_n: we consider as a representative of the class $A_{i,j}$ a braid that restricts to the trivial $(n-1)$-braid if the i-th string is removed, and such that the i-th string loops once around the j-th string after passing above all the others (see again Figure 1.12). Therefore we can write

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \tag{1.26}$$

for $1 \leq j \leq n$.

Consider a pure braid $A \in P_n$. Suppose we remove the first string and, via an isotopy of the unit cylinder, move all the others in such a way that if we replace the removed string with a vertical straight line, it does not intertwine with the other components of A. We call A_1 the $(n-1)$-braid obtained from A by removing the first string and B the result after the substitution. Let $C = AB^{-1}$. Then C has a peculiar property: by removing the first string we get $A_1A_1^{-1} = 1_n$. If we take along the first string during this "combing" operation, it will be entangled with the other strings (perhaps even in a very complicated way) and the result will look somewhat like Figure 1.15, where is shown the braid C after this pull and untie operation.

Definition 1.27. An n-braid is said to be i-pure if, removing the i-th string, one gets the $(n-1)$-trivial braid.

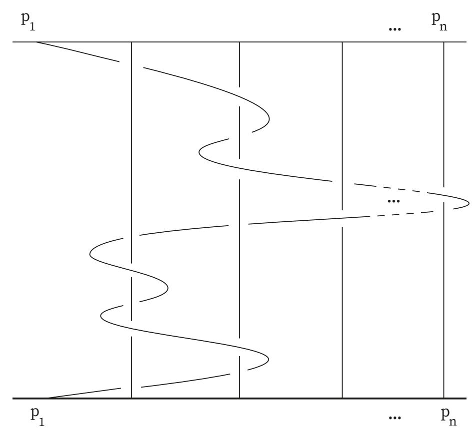


Figure 1.15: the result after the "combing" operation

We had $C = AB^{-1}$, so $A = CB$. Hence, A is the product of a 1-pure braid and a braid B obtained as described above.

Iterating this procedure on the other strings of B starting from the second one, we get:

$$A = C_1 C_2 \dots C_{n-1},$$

where in every C_i all strings but the i-th are vertical straight lines, and the i-th one is only involved in crossings with strings of higher index. It can also be proved ([1]) that this expression uniquely determines A. This argument represents an intuitive proof of the following:

Lemma 1.28. *If $\beta \in P_n$, then it is uniquely expressible as a product $\beta = b_{n-1}b_{n-2}\cdots b_1$, where each pure n-braid b_i is a reduced word in the free group $F(A_{i,i+1}, \dots, A_{i,n})$ with generators $A_{i,i+1}, \dots, A_{i,n}$. The expression $\beta = b_{n-1}b_{n-2}\cdots b_1$ is called the normal form for β.*

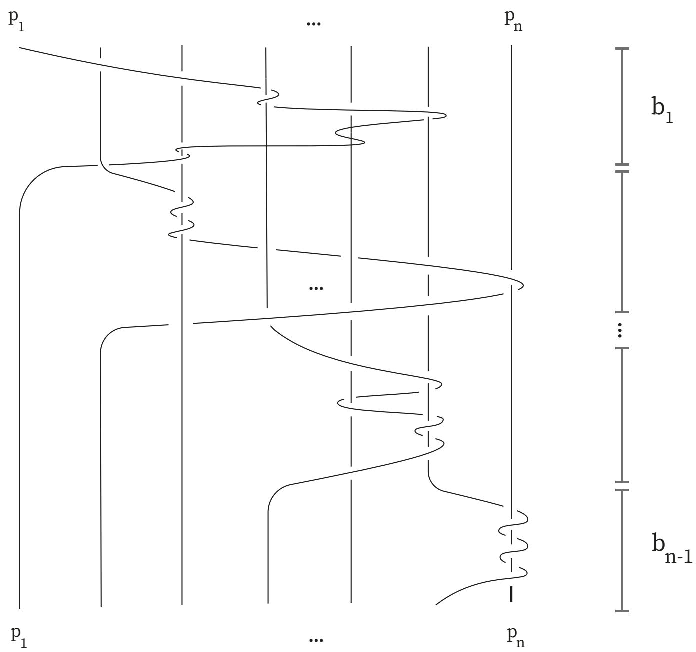


Figure 1.16: An example of normal form of a braid

Another important result which we will need in the next paragraphs is a representation of the braids group, due to Artin ([1]). Let $\mathrm{Aut}(F_n)$ be the group of all automorphisms of F_n, then B_n can be represented as a subgroup of $\mathrm{Aut}(F_n)$. The representation $\rho : B_n \to \mathrm{Aut}(F_n)$ is defined as

$$\rho(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \qquad j \neq i, i+1. \end{cases} \tag{1.29}$$

Theorem 1.30. *(Artin) The Artin representation of the braids group is faithful.*

Proof. See [1, Theorem 14]. □

1.3 Homotopy braids group

Let us move on to the discussion of braids up to homotopy. Recall from the definition that if I_i is a copy of the closed unit interval and β is an n-braid, we have that $\beta_i : I_i \to D \times I$ is a parametrization of the i-th string of β for each $i = 1, \dots, n$.

A homotopy between β_i and β'_i is a continuous map

$$H_i : I_i \times [0,1] \to D \times I$$

such that $H_i(x, 0) = \beta_i(x)$ and $H_i(x, 1) = \beta'_i(x)$ for all $x \in I_i$. In what follows we should denote $H_i(x,t)$ by $h_{i,t}(x)$. According to [5], we have the following:

Definition 1.31. Two braids are *homotopic* if one can be deformed to the other by simultaneous homotopies of the braid strings in the unit cylinder that fix the endpoints, so that different strings never intersect.

In other words, two n-braids β and β', parametrized respectively by the proper embeddings $(\beta_1, \dots, \beta_n)$ and $(\beta'_1, \dots, \beta'_n)$ are homotopic if there exist homotopies H_i between the maps β_i and β'_i, for $i = 1, ..., n$, so that the sets $h_{1,t}(I_1), \dots, h_{n,t}(I_n)$ are disjoint for each value of $t \in [0,1]$. Hence, in a homotopy between two braids self-intersections of strings are allowed.

In this context, we still assume that the braids are oriented from top to bottom and that they are monotone. However, we relax this constraint during the homotopic transformation, that is, we don't require that homotopies between two braids are level-preserving. As well as braids isotopy, also the relation of homotopy is reflexive, symmetric and transitive. Moreover, the operation of braids composition defined in 1.2 induces a group multiplication on the set of homotopy classes of n-braids.

In [5], D.L. Goldsmith proved that the notions of braid isotopy and braid homotopy are distinct. In fact, denote by $\tilde{B}_n$ the so-called *homotopy n-braid group* and define the groups homomorphism

$$I_n : B_n \to \tilde{B}_n$$

which maps the isotopy class of each braid to its homotopy class. Figure 1.17 demonstrates a homotopy from an isotopically non-trivial braid to the trivial braid and hence shows that $\ker(I_n) \neq 0$.

Remark 1.32. *We have that* $\ker(I_n) \subset P_n$*. Indeed, let* $\beta \in B_n$ *be such that* $I_n(\beta)$ *is a homotopically trivial braid. Then clearly it induces the identity permutation, namely* β *is a pure* n*-braid.*

In order to derive a presentation of the group $\tilde{B}_n$ we need a few more preliminary lemmas.

Lemma 1.33. *A pure* n*-braid* β *is in* $\ker(I_n)$ *if and only if each factor of its normal form is in* $\ker(I_n)$*.*

Proof. Being I_n a homomorphism, only the necessary condition is non-trivial. Let $\beta \in \ker(I_n)$. Note that if we remove the first k strings from β, the resulting $(n-k)$-braid is homotopic to the trivial one. This $(n-k)$-braid is a representative of the class $b_{n-1}b_{n-2}\cdots b_k \in \ker(I_n)$, $k = 0, 1, \ldots, n-1$. Hence, for $k = n-1$ we have that $b_{n-1} \in \ker(I_n)$. Now let $0 \leq k < n-1$ and assume that $b_i \in \ker(I_n)$ for each $k < i \leq n-1$. We had that $b_{n-1}b_{n-2}\cdots b_k \in \ker(I_n)$: by our assumption and since $\ker(I_n)$ is a subgroup, we can write $b_{n-1}b_{n-2}\cdots b_{k+1} \in \ker(I_n)$. Then $(b_{n-1}b_{n-2}\cdots b_{k+1})^{-1}(b_{n-1}b_{n-2}\cdots b_k) = b_k \in \ker(I_n)$. $\square$

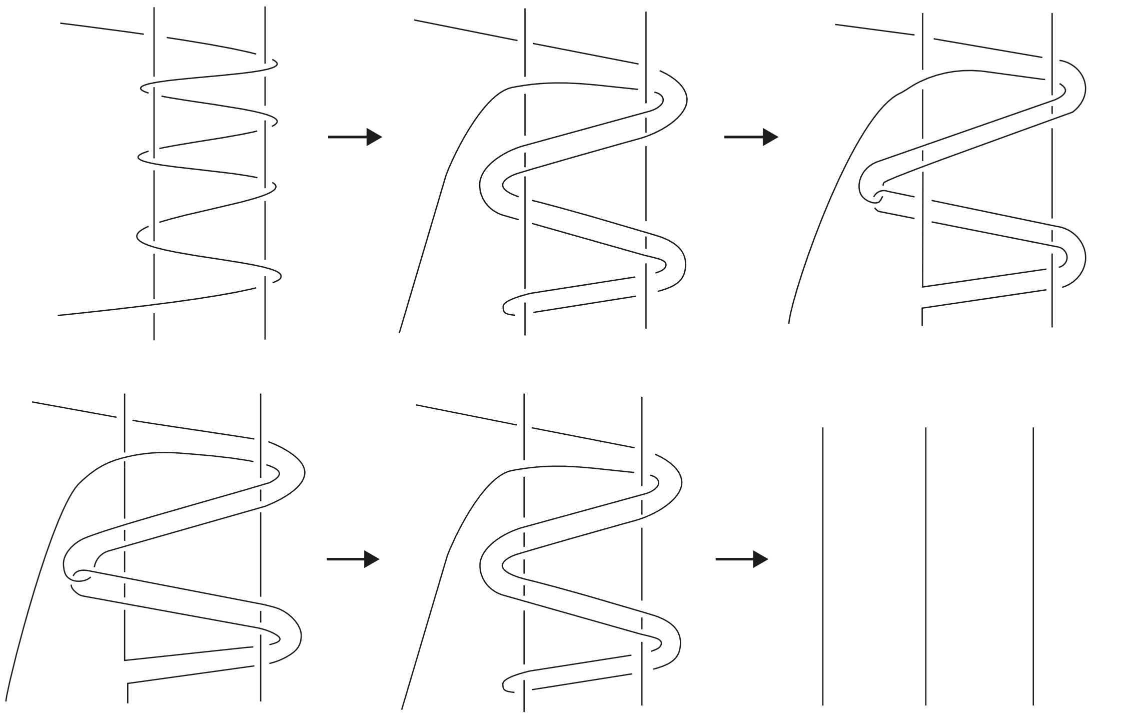


Figure 1.17: A homotopy between $A_{1,2}^{-1}(A_{1,3}^{-1}A_{1,2}^{-1}A_{1,3})A_{1,2}(A_{1,3}^{-1}A_{1,2}A_{1,3}) \in B_3$ and 1_3

Lemma 1.34. *Let $\beta \in P_n$ be a reduced word in the free group $F(A_{i,i+1}, \ldots, A_{i,n})$. Then β is in $\ker(I_n)$ if and only if β is in the normal subgroup of P_n generated by $\left[A_{i,j}, gA_{i,j}g^{-1}\right]$, $1 \leq i \leq n-1$, $2 \leq j \leq n$, $i < j$ and $g \in F(A_{i,i+1}, \ldots, A_{i,n})$.*

Proof. Assume that $\beta \in \ker(I_n)$. Delete the first $i-1$ strings of β and identify initial and terminal endpoints of the remaining $(n-i+1)$-braid, forming a closed braid which gives a link L in $\mathbb{R}^3$; denote by ℓ_k the component of L that comes from the k-th string of β. By hypothesis, L is homotopic to the trivial link, in the sense of J. Milnor ([16]). Then, by [16, Corollary 1], ℓ_i represents the identity in the link group $\mathscr{G}(L')$ of $L' = \ell_{i+1} \cup \cdots \cup \ell_n$. Thus ℓ_i represents an element in a certain normal subgroup of $\pi(\mathbb{R}^3 \setminus L')$. Recall that since L' is a trivial link, $\pi(\mathbb{R}^3 \setminus L')$ is the free group $F(y_{i+1}, \ldots, y_n)$ and the normal subgroup mentioned above is the group generated by $\left[y_k, xy_kx^{-1}\right]$, $k = i+1, \ldots n$, and $x \in F(y_{i+1}, \ldots, y_n)$.

Hence β must be in the normal subgroup of P_n generated by $[A_{i,j}, gA_{i,j}g^{-1}]$, $1 \leq i \leq n-1$, $2 \leq j \leq n$, $i < j$ and $g \in F(A_{i,i+1}, \ldots, A_{i,n})$. The proof of the converse is essentially contained in Figure 1.17, in which it is explained how a particular generator of this subgroup - $\sigma_{1,2}^{-1}(\sigma_{1,3}^{-1}\sigma_{1,2}^{-1}\sigma_{1,3})\sigma_{1,2}(\sigma_{1,3}^{-1}\sigma_{1,2}\sigma_{1,3})$ - is represented by a homotopically trivial braid. Finally, notice that if $\tilde{\beta}$ is a generator of this subgroup and L is the closed braid formed from $\tilde{\beta}$, the homotopy which Milnor constructs in [16] taking L to the trivial link gives a homotopy from $\tilde{\beta}$ to the trivial braid. $\square$

Remark 1.35. *The normal subgroup described in Lemma 1.34 is exactly the reduced quotient of* $F(A_{i,i+1}, \ldots, A_{i,n})$.

Example 1.36. Let $n = 4$, $i = 1$, $j = 4$, $g = A_{12}A_{13} \in F(A_{12}, A_{13}, A_{14})$. Figure 1.18 shows the commutator $\beta = \left[A_{14}^{-1}, (A_{12}A_{13})^{-1}A_{14}^{-1}(A_{12}A_{13})\right]$

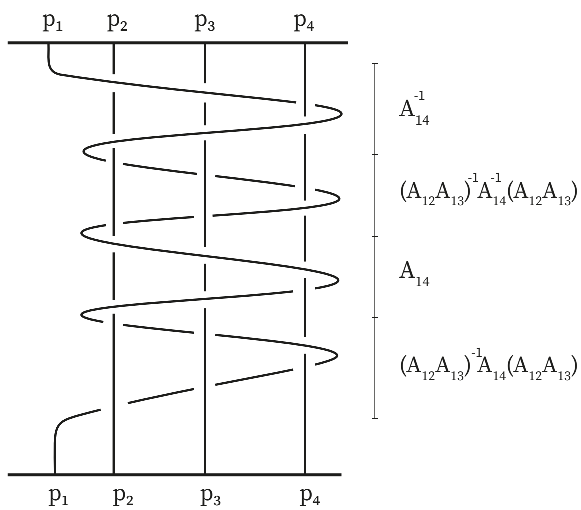


Figure 1.18

As a consequence of Lemmas 1.28,1.33,1.34 we have

Theorem 1.37. *(Goldsmith)*

$$\ker(I_n) = \left\langle\!\left\langle \left[A_{i,j}, gA_{i,j}g^{-1}\right] \mid 1 \leq i \leq n-1,\ 2 \leq j \leq n,\ i < j,\ g \in F(A_{i,i+1}, \dots, A_{i,n}) \right\rangle\!\right\rangle$$

Proof. See [5]. □

From the results above we can thus derive a presentation of the homotopy braid group.

Corollary 1.38. *The homotopy n-braid group $\tilde{B}_n$ is isomorphic to the n-braid group B_n with the additional relations*

$$[A_{i,j}, gA_{i,j}g^{-1}] = 1_n,$$

for any $1 \leq i \leq n-1$, $2 \leq j \leq n$, $i < j$ and $g \in F(A_{i,i+1}, \dots, A_{i,n})$.

Proof. Notice that I_n is surjective. The proof is a consequence of Theorem 1.37 and the first isomorphisms theorem. □

We have seen that

$$B_n \cong \left\langle \sigma_1, \dots, \sigma_n \;\middle|\; \begin{array}{cl} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |j-i| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \dots, n-2. \end{array} \right\rangle$$

and, by Theorem 1.37,

$$\tilde{B}_n \cong {}^{B_n}\!/_{\ker(I_n)} \cong {}^{B_n}\!/_{J}.$$

where $J \coloneqq \left\langle\!\left\langle \left[A_{i,j}, gA_{i,j}g^{-1}\right],\ 1 \leq i \leq n-1,\ 2 \leq j \leq n,\ i < j,\ g \in F(A_{i,i+1}, \dots, A_{i,n}) \right\rangle\!\right\rangle$.

As we did for the braids group B_n, we can derive a presentation of the homotopy braids group $\tilde{B}_n$. If we denote by $\tilde{\rho} : \tilde{B}_n \to \mathrm{Aut}RF_n$ the so-called *homotopic Artin representation*, defined by the same expression of 1.29, we have:

Theorem 1.39. *The homotopic Artin representation is faithful.*

Proof. This result is stated and proved in [7, Theorem 1.7]. □

Being the homotopy relation weaker than isotopy, Alexander's theorem still holds if we consider links up to link-homotopy: for every link $L \in \mathbb{R}^3$, we can always find $\beta \in \tilde{B}_n$ for some $n \in \mathbb{N}$ such that $\hat{\beta}$ and L are link-homotopic. Moreover, Habegger and Lin give sufficient and necessary conditions for two homotopy braids to have link-homotopic closures as oriented links, thus describing a Markov-type theorem up to homotopy. In particular, braids stabilisation is no longer needed and it is replaced by the so called *partial conjugation* (for details, see [7, Theorem 2.13]).

Chapter 2

Claspers: a short introduction

2.1 Definition and properties

We briefly summarize how Habiro ([8]) uses special surgeries encoded by certain surfaces, that he calls claspers, in order to introduce the C_k-equivalence between links, an equivalence relation that arises from claspers in the sense of the next definitions. We will denote with M a smooth, compact, connected and oriented 3-manifold.

Definition 2.1. Let A_1, A_2 be two annuli connected by a band B. We define a *basic clasper* $C \subset M$ as the image in M via an embedding of the non-oriented, planar surface $A_1 \cup A_2 \cup B$. We call A_1, A_2 and B respectively the *leaves* and the *edge* of C.

Consider a basic clasper C and a small regular neighbourhood N_C of C in M. Then, we can associate to C a framed link L_C, unique up to isotopy, as follows. Make a crossing change between the annuli along B as in figure 2.1(a). The framed link $L_C = L_{C,1} \cup L_{C,2}$, here drawn in the blackboard framing convention, is determined by A'_1 and A'_2 as illustrated in figure 2.1(b).

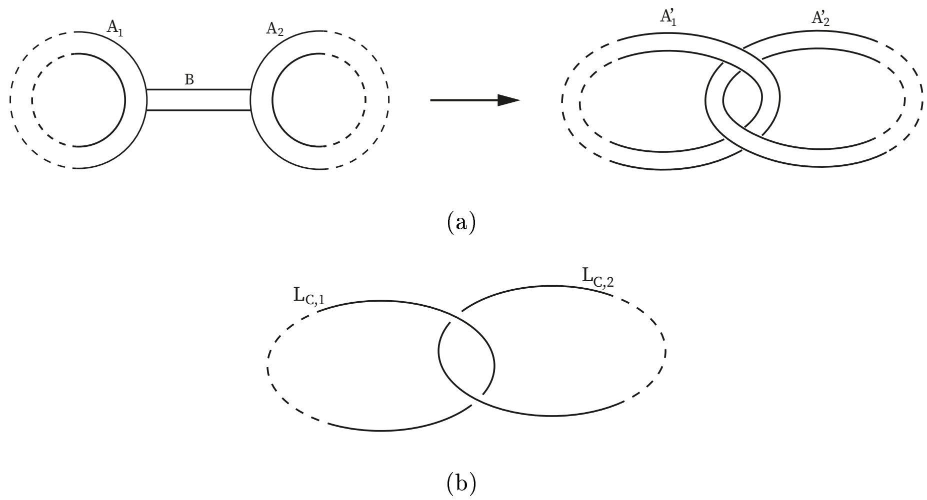


Figure 2.1: how to associate a framed link to a basic clasper

A different choice for the crossings in figure 2.1(a) lead to different framed links.

We define *surgery on the basic clasper* C to be the surgery on the associated framed link L_C. In this section we will denote by M^C the resulting 3-manifold obtained from M by surgery on C.

The following result contains the starting point for most of properties of claspers.

Proposition 2.2. *[8, Proposition 2.2]*

1. *Let $C = A_1 \cup A_2 \cup B$ be a basic clasper in a 3-manifold M, and D a disk embedded in M such that A_1 is a collar neighbourhood of ∂D in D and $D \cap C = A_1$. Let N be a small regular neighbourhood of $C \cup D$ in M (a solid torus). Then there is a diffeomorphism*

$$\phi_{C,D}{}_{|_N} : N \xrightarrow{\cong} N^C$$

fixing $\partial N = \partial N^C$ pointwise, which extends to a diffeomorphism $\phi_{C,D} : M \xrightarrow{\cong} M^C$ restricting to the identity on $M \setminus \operatorname{int} N$.

2. *In* (1) *assume that there is a parallel family* X *of "objects" (eg, links, claspers, etc.) transversely intersecting the open disk* $D \setminus A_1$ *as depicted in Figure 2.2(a). Then* ${\phi_{C,D}}^{-1}(X^C)$ *in* M *looks as depicted in figure 2.2(b).*

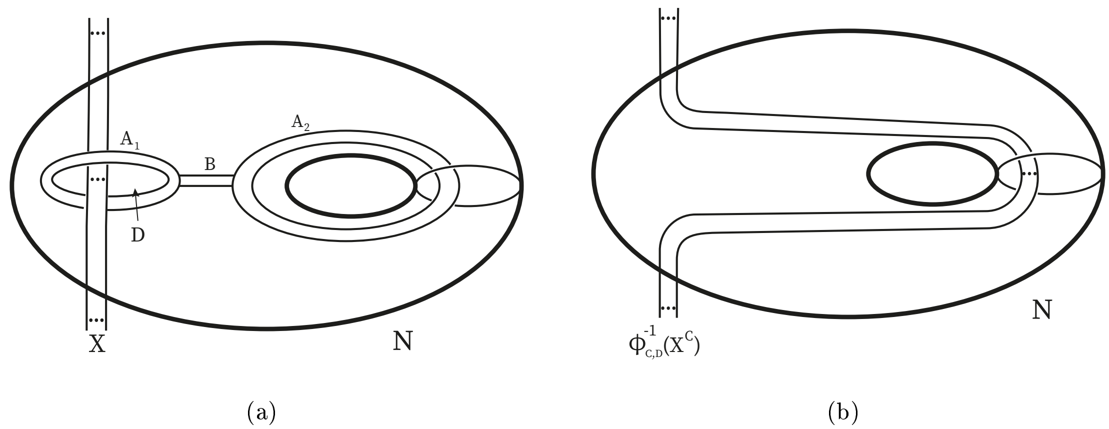


Figure 2.2: effect of surgery on the clasper $C = A_1 \cup A_2 \cup B$

Proof. 1. Let $L_C = L_{C,1} \cup L_{C,2} \subset N$ be the framed link associated to C. In this construction the component $L_{C,1}$ bounds a disk D' in $\mathrm{int}(N)$ intersecting $L_{C,2}$ transversely once, and $L_{C,1}$ has trivial framing: so, there is a diffeomorphism

$$\phi_{C,D_{|N}} : N \overset{\cong}{\to} N^{L_C} (= N^C)$$

that restricts to the identity on ∂N.

2. Here L_C looks as depicted in Figure 2.3(a). Before performing surgery on L_C, slide the object X along the component $L_{C,2}$ obtaining an object X' in M depicted in Figure 2.3(b). In this situation, Dehn surgery is equivalent (up to diffeomorphism) to discard L_C, hence $\phi^{-1}_{C,D}(X^C) \subset M$ looks as shown in Figure 2.2(b). For more details, see [8, Proposition 2.2].

□

Notice that if M is a 3-ball or a 3-sphere, then $\phi_{C,D}$ does not depend on D and we will denote it by ϕ_C.

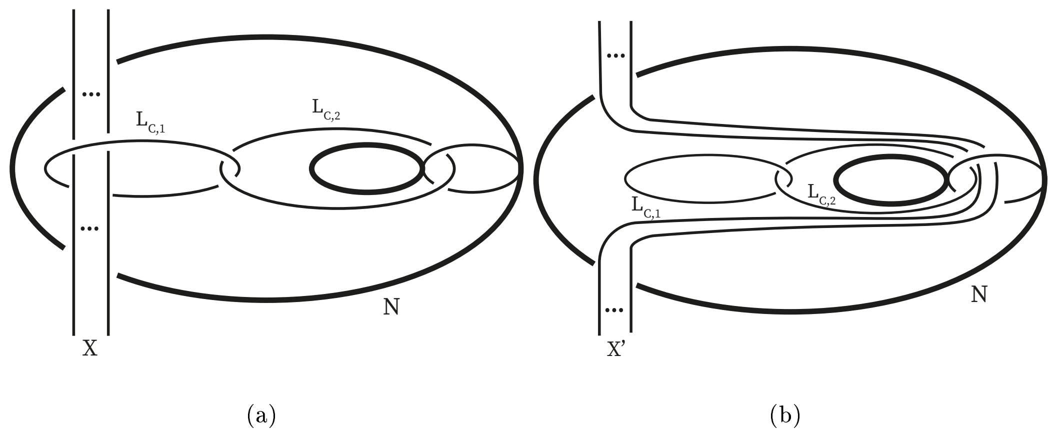


Figure 2.3: Proof of Proposition 2.2

Remark 2.3. *As a special case of Proposition 2.2, consider a basic clasper* $C = A_1 \cup A_2 \cup B$ *such that both annuli are crossed by two parallel families of strings in a link* γ. *In this situation, surgery on* C *produces a "clasp" of the two parallel sets of strings. This is the reason for the name clasper.*

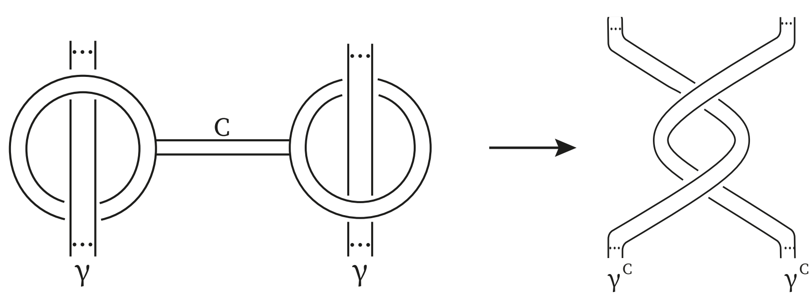


Figure 2.4

Definition 2.4. A clasper $G = A \cup B$ for a link γ in a 3–manifold M is a non-oriented compact surface embedded in the interior of M which decomposes in two subsurfaces A and B. Connected components of A are called *constituents* of G, while those of B are the *edges*. Each edge is a band disjoint from γ connecting two (not necessarily distinct) constituents. There are four kinds of constituents: *leaves*, *disk-leaves*, *nodes* and *boxes*. Disk-leaves, nodes and boxes are disks, while the leaves are annuli. All of them are disjoint from γ, except for the disk-leaves, that may intersect γ transversely. Furthermore, the constituents must satisfy the following conditions:

1. each node has three incident ends (an end is one of the two components of the intersection between A and an edge of G);
2. each leaf has just one incident end;
3. each box R of G has three incident ends, one of which is distinguished from the other two (see figure 2.5(c)).

A disk-leaf of a clasper (for a link γ) is called *trivial* if it does not intersect γ and *simple* if it intersect γ in a single point.

Definition 2.5. A clasper G for a link γ is *simple* if all its disk-leaves are simple.

Now, consider a clasper G and replace nodes, disk-leaves and boxes with some leaves as in figure 2.5. This is how we can associate to G a clasper C_G consisting of all basic claspers. In particular, the number of basic claspers in C_G is equal to the number of edges in G. Finally, we define the *surgery on a clasper* G as the surgery on the associated clasper C_G. If N_G is a regular neighbourhood of G in M we can write:

$$M^G = \left(M \setminus \operatorname{int} N_G\right) \bigcup_{\partial N_G} {N_G}^{C_G}$$

where M^G is the result from M of the surgery on G. Hence, if a regular neighbourhood

N_G of G is specified we can identify $M \setminus \operatorname{int} N_G$ with $M^G \setminus \operatorname{int} {N_G}^G$.

Before going on, we need a couple of basic properties of claspers; as for links, we can give the notion of *tame* clasper. Recall that a *handlebody* of genus g is the result of attaching g disjoint 1-*handles* $D^2 \times [-1, 1]$ to a $3-$ball B^3 by sewing the parts $D^2 \times \{\pm 1\}$ to $2g$ disjoint disks on ∂B^3 in such a way that the result is an orientable $3-$manifold with boundary. Two handlebodies of the same genus are diffeomorphic and vice versa ([19, 2.G.14]).

Definition 2.6. Let $V = V_1 \cup \cdots \cup V_n$ be a disjoint union of handlebodies in the interior of a 3-manifold M, γ a link trasnverse to ∂V and $G \subset \operatorname{int} V$ a clasper for γ. We say that G is *tame* in V if there is an orientation-preserving diffeomorphism $\Phi_{(V,G)_{|V}} : V \xrightarrow{\cong} V^G$ that restricts to the identity on ∂V.

If G is a tame clasper, then the diffeomorphism $\Phi_{(V,G)_{|V}}$ extends to

$$\Phi_{(V,G)_{|V}} : M \xrightarrow{\cong} M^G (= M \setminus \operatorname{int} V \cup V^G)$$

which restricts to the identity outside V. We denote by $\gamma^{(V,G)}$, or simply γ^G, the link ${\Phi_{(V,G)}}^{-1}(\gamma^G)$ in M and call it the result from γ of surgery on the pair (V, G) or, more often, simply on G. Notice that surgery on (V, G) transforms a link in M into another link in M.

Finally, consider two pairs of links and tame claspers (γ, G), (γ', G') in a 3-manifold M. We will write $(\gamma, G) \sim (\gamma', G')$ if the results of surgeries γ^G and $\gamma'^{G'}$ are diffeomorphic. In [8, Proposition 2.7], Habiro gives sufficient conditions (more precisely, moves on the diagrams of G and G') for γ^G and $\gamma'^{G'}$ to be diffeomorphic.

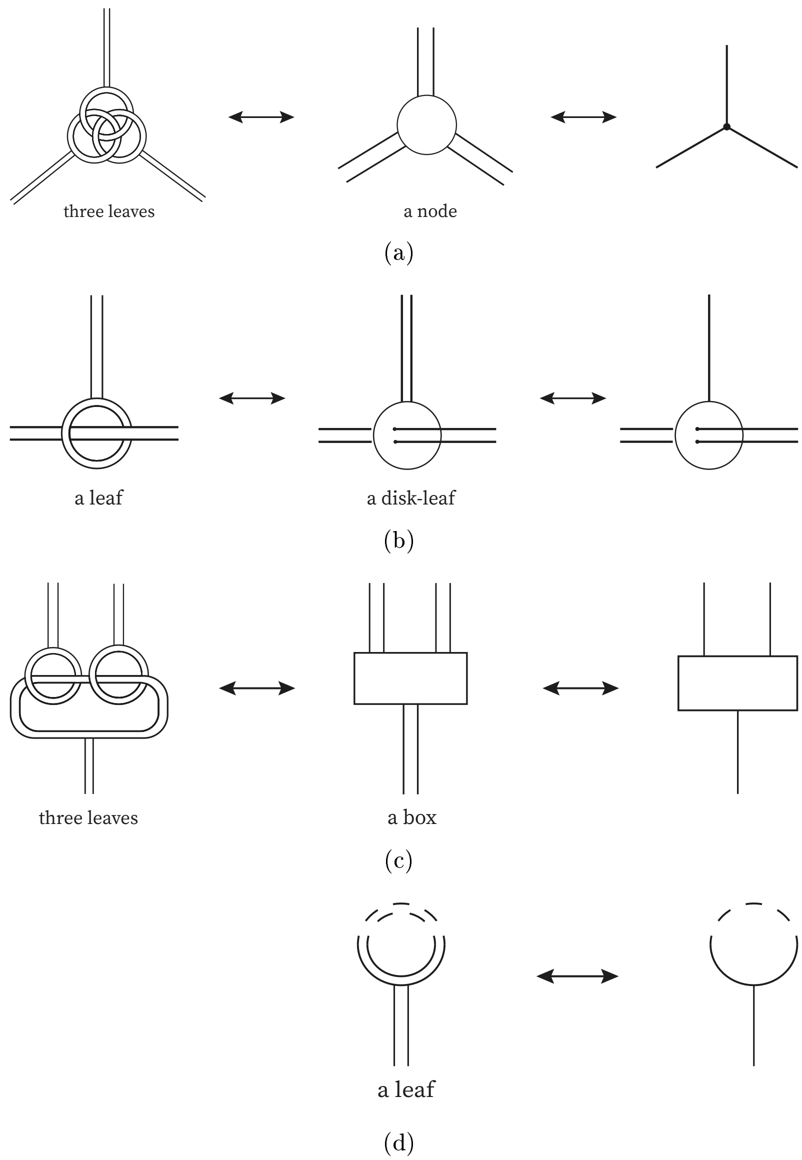


Figure 2.5: Possible constituents of a clasper, with their representation in the blackboard framing convention and how to replace them with leaves.

2.2 C_k−equivalence relations on links

Definition 2.7. A *tree clasper* T for a link γ in a 3-manifold M is a connected clasper without boxes and such that the union of the nodes and edges of T is simply connected.

An example of tree clasper for a link is given in Figure 2.6. We also say that a tree clasper is *admissible* if it has at least one disk-leaf and *strict* if (moreover) it has no leaves. Notice that, having no boxes and no leaves, the surface defined by a strict tree clasper is diffeomorphic to the disk D^2.

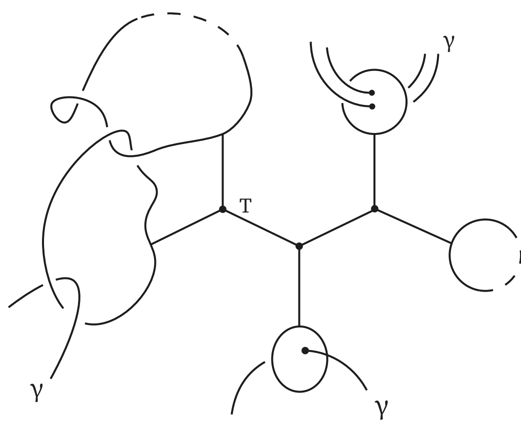


Figure 2.6: A tree clasper T for a link γ

Proposition 2.8. *Every admissible tree clasper for a link in a 3-manifold M is tame. In particular , every strict tree clasper is tame.*

Proof. See [8, Proposition 3.3]. □

It follows that an admissible tree clasper T for a link γ in a 3-manifold M determines a link γ^T in M. Hence, by tameness, surgery on an admissible tree clasper can be seen as an operation on links in a *fixed* 3-manifold M. Another important property of tree claspers is given by the following

Proposition 2.9. *Let T be an admissible tree clasper for a link γ in M with at least one trivial disk-leaf. Then γ^T is equivalent to γ.*

Proof. In [8, Proposition 3.4] Habiro shows that there exists a sequence of admissible forest claspers (i.e, a finite union of admissible tree claspers) for γ, say $G_0 = T, \ldots, G_p = \emptyset$, $(p \geq 0)$, from T to $\emptyset$ such that for each $i = 0, \ldots, p-1$, G_{i+1} is obtained from G_i by claspers moves (see [8, Proposition 2.7]). Hence we get $\gamma = \gamma^{\emptyset} = \gamma^{G_p} \cong \gamma^{G_0} = \gamma_T$. $\square$

Definition 2.10. Define the *degree* $\deg T$ of a strict tree clasper T for a link γ as the number of its nodes plus 1.

Definition 2.11. Let M be a 3-manifold and $k \geq 1$ an integer. A (*simple*) *C_k-move* on a link γ in M is a surgery on a (simple) strict tree clasper of degree k. More precisely, we say that two links γ and γ' in M are related by a (simple) C_k-move and write

$$\gamma \underset{C_k}{\longrightarrow} \gamma' \qquad \left(\gamma \underset{sC_k}{\longrightarrow} \gamma'\right)$$

if there is a (simple) strict tree clasper T for γ with $\deg T = k$, such that γ^T is equivalent to γ'.

We are now able to define the *C_k-equivalence* (*sC_k-equivalence*) as the equivalence relation on links generated by C_k-moves (simple C_k-moves) and ambient isotopies.

Remark 2.12. *It's easy to check that the C_k-equivalence is reflexive and transitive, while the symmetry is ensured by [8, Proposition 3.23]. Moreover, the C_k-equivalence relation becomes finer as k increases, i.e. C_l-equivalence implies C_k-equivalence for every $l \geq k \geq 1$.*

Example 2.13. As seen in Proposition 2.2 and Figure 2.2, a simple C_1-move is equivalent to a crossing change. It is also equivalent to band-summing a Hopf link L_2, see Figure 2.7(a). Hence any two knots in $\mathbb{S}^3$ are C_1-*equivalent* to each other.

A simple C_2-move is equivalent to band-summing the Borromean rings L_3, as in Figure 2.7(b). In [17] is proved that any two knots in $\mathbb{S}^3$ are related by a sequence of operations of this kind, which are called "$\Delta-$unknotting operations".

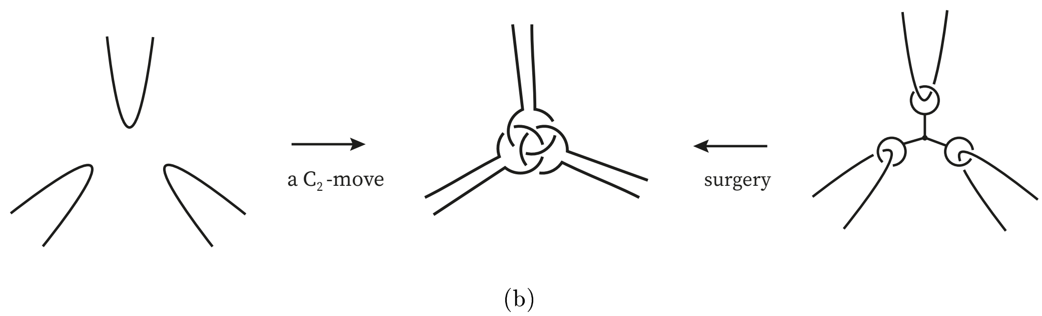


Figure 2.7: Examples of C_k moves for $k = 1, 2$.

We conclude this section with a conjecture posed by Habiro.

Definition 2.14. Two links in M are said to be C_∞-*equivalent* if they are C_k-equivalent for all $k \geq 1$.

Conjecture 2.15. Two links in a 3-manifold M are equivalent if and only if they are C_∞-equivalent.

2.3 Clasper calculus up to link homotopy

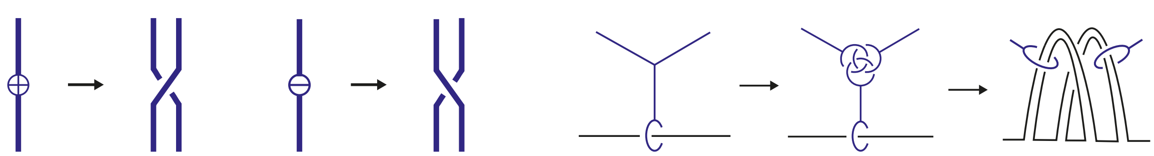
The previous sections give some insight into how claspers can be a powerful tool to deal with link-homotopy. Let us continue with the discussion on their properties, especially for what concerns homotopy braids.

Recall that M is a smooth, compact, connected and oriented 3-manifold and denote by X an n-component ordered and oriented 1-manifold (i.e. a disjoint union of n circles and intervals).

Definition 2.16. An n-component *tangle* θ in M is the image of a smooth embedding of X into M with the induced orientation.

Once again, we say that two tangles are isotopic if they are related by an ambient isotopy of M that fixes the boundary. Moreover, we say that two tangles are link-homotopic if there is a homotopy between them fixing the boundary, and such that distinct components remain disjoint during the deformation.

In what follows, we will focus on a special type of clasper: unless otherwise specified, by a clasper for a tangle θ we mean an admissible and strict tree clasper T for θ with no trivial disk-leaves. In figures we indicate how the edges of the claspers are twisted using markers called *twists* as in Figure 2.8(a). In Figure 2.8(b) we see first how to replace a node with 3 leaves and then we apply Proposition 2.2 to get rid of one of them.



(a) Local diagrammatic representation of twists

(b) An example of surgery rule

Figure 2.8

Definition 2.17. Let T be a clasper for a tangle θ. The *support* of T, denoted $\mathrm{supp}(T)$, is the set of the components of θ that intersect T.

Remark 2.18. *We will often consider the number of the components rather than the components themselves.*

Recall from 2.1 that if T is a simple clasper for a tangle θ, then all its leaves intersect θ exactly once: a leaf of a simple clasper intersecting the l-th component is called an l-leaf.

Definition 2.19. We say that T *has repeats* if it intersect a component of θ in at least two points.

In [8] Habiro developed some operations on tangles that allow to deform one clasper into another with isotopic surgery result. In what follows, we try to explain the analogous calculus up to link-homotopy, as described in [6].

Definition 2.20. Let T and S denote two simple claspers for a given tangle θ. We say that *T and S are link-homotopic*, and we write $T \sim S$, if the surgery result θ^T and θ^S are so. Moreover, if θ^T is link-homotopic to θ, we say that T *vanishes up to link-homotopy* and we denote $T \sim \emptyset$.

A special case ($k = 1$) of [4, Lemma 1.2] gives us the following

Lemma 2.21. *Let T be a simple clasper for a tangle. If T has repeats then it vanishes up to link-homotopy.*

Now let T a clasper for a tangle θ. A clasper T' is a parallel copy of T if it has same support, same constituents and the two planar projections are parallel to each other. Moreover, we choose crossings in such a way, in their diagrams, T is always above T'. Combining Lemma 2.21 with the proofs of Habiro's result on clasper calculus yields the following link-homotopy equivalences, represented in Figure 2.9.

Corollary 2.22. *Let T be a simple clasper for a tangle θ:*

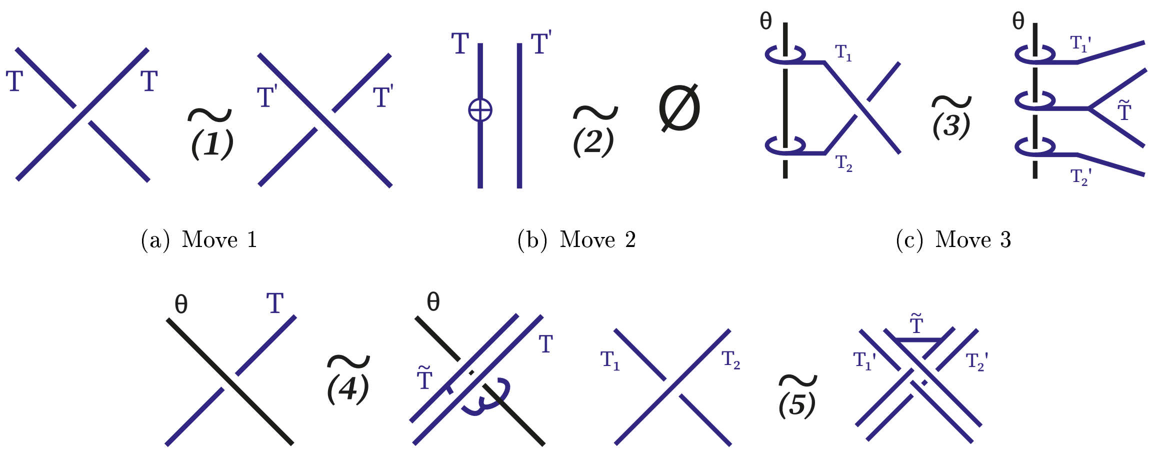
1. *If T' is obtained from T by a crossing change between two edges of T (Figure 2.9(a)) then $T \sim T'$.*

2. *If T' is a parallel copy of T which differs from T only by one twist (Figure 2.9(b)), then $T \cup T' \sim \emptyset$.*

3. *If T_1 and T_2 have two adjacent leaves and if $T_1' \cup T_2'$ is obtained from $T_1 \cup T_2$ by exchanging these leaves as depicted in Figure 2.9(c), then $T_1 \cup T_2 \sim T_1' \cup T_2' \cup \tilde{T}$, where $\tilde{T}$ is as drawn in the figure.*

4. *If T' is obtained from T by a crossing change with a strand of θ as in Figure 2.9(d), then $T \sim T' \cup \tilde{T}$, where $\tilde{T}$ is as drawn in the figure.*

5. *If $T_1' \cup T_2'$ is obtained from $T_1 \cup T_2$ by a crossing change between one edge of T_1 and one of T_2 (Figure 2.9(e)), then $T_1 \cup T_2 \sim T_1' \cup T_2' \cup \tilde{T}$, where $\tilde{T}$ is as drawn in the figure.*



(a) Move 1 (b) Move 2 (c) Move 3

(d) Move 4 (e) Move 5

Figure 2.9: Link-homotopy claspers equivalences

Proof. This results are summarized in [13, Lemmas 2.4, 2.5, 2.9, 2.10 and 2.12]. The key observation is that the results of [8] used here are up to C_k-equivalence; by construction, all such higher degree claspers have the same support as the initial ones and hence they must have repeats; Lemma 2.21 allows us to neglect those higher degree claspers up to link-homotopy. □

Remark 2.23. *Statement (5) implies that if* $|supp(T) \cap supp(T')| \geq 1$ *then we can make crossing changes between the edges of* T *and* T'*. Moreover, if* $|supp(T) \cap supp(T')| \geq 2$ *we can also exchange the leaves of* T *and* T' *thanks to statement (3). Finally, statement (4) allows crossing changes between* T *and a component of* θ *in the support of* T*. In fact, in all cases the clasper* $\tilde{T}$ *involved in the corresponding statements has repeats and can thus be deleted up to link-homotopy.*

The next lemma from [13, Lemmas 2.6, 2.7, 2.8] describes how to handle twists.

Lemma 2.24. *Let* T *be a simple clasper for a tangle. Then:*

6. *If* T' *is obtained from* T *by turning a positive twist into a negative one then* $T \sim T'$*.*
7. *If* T' *is obtained from* T *by moving a twist across a node then* $T \sim T'$*.*
8. *If* T *and* T' *are identical outside a neighbourhood of a node, and if in this neighbourhood* T *and* T' *are as depicted in the last picture of Figure 2.10, then* $T \sim T'$*.*

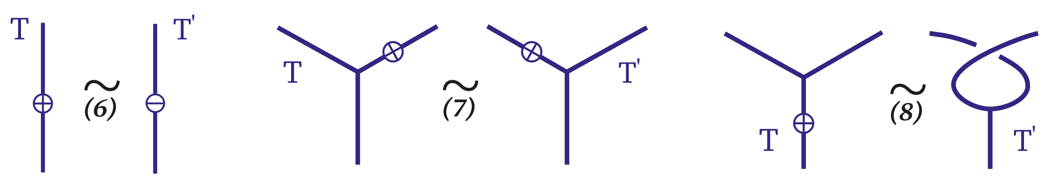


Figure 2.10: Moves (6), (7) and (8) of Lemma 2.24

Proof. For statement *(6)*, consider the union $T \cup \tilde{T} \cup T'$, where $\tilde{T}$ is another parallel copy between T and T' without twists. By *(2)* from 2.22, this union is both link-homotopic to T and T'. We can proceed in the same way for statement *(7)*. Statement *(8)* is achieved by an isotopy (Figure 2.11). □

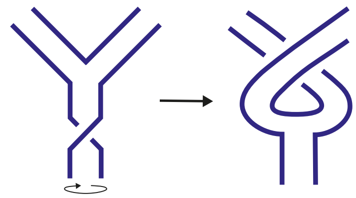


Figure 2.11: Proof of move (8) from Lemma 2.24

Remark 2.25. *Notice that by moves (6)-(7) of Lemma 2.24 we are allowed to bring all the twists on the same edge and then cancel them pairwise. Thus we can consider, up to link-homotopy, only claspers with one or no twists, called respectively twisted and untwisted claspers.*

Another important result which we will need is the so called *IHX relation* for claspers (see for instance [13, Lemma 2.11]):

Proposition 2.26. *Let T_I, T_H, T_X be three parallel copies of a given simple clasper that coincide everywhere outside a 3-ball, where they are as shown in Figure 2.12. Then we have that $T_I \cup T_H \cup T_X \sim \emptyset$.*

Proof. A detailed proof is contained in [9, Theorem 6.2]. □

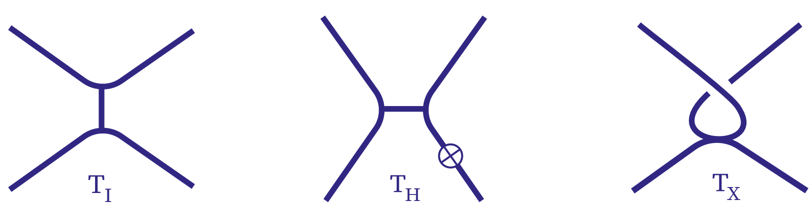


Figure 2.12: The IHX relation for claspers

Chapter 3

Braids and claspers

3.1 Comb-claspers

We have introduced most of the tools we will need to interpret homotopy braids through clasper calculus. In the next sections we follow [6] to define a linear and faithful representation of the homotopy braid group based on the notion of *comb-claspers*.

Observe that braids are tangles in $D \times I$ without closed components, but with monotonic and boundary conditions: initial and final endpoints are fixed and coincide with the sets of points $\{(p_i, 1)\}_{i=1}^n$, $\{(p_i, 0)\}_{i=1}^n$ as specified in Section 1.2. Moreover, any tangle with the same boundary condition and without closed components (also called a *string link*) is link-homotopic to a braid. This fact is proved in [7, Lemma 1.8] in the pure case, i.e. when each component of the string link connects each point $(p_i, 1)$ with the correspondent $(p_i, 0)$. We revisited it here from a more geometric and intuitive point of view in both pure and general cases.

Proposition 3.1. *Every (pure) string link is link-homotopic to a (pure) braid.*

Proof. We start from the pure case. Let θ be a pure n-string link and proceed by induction on the number $n \geq 2$ of strings. The case $n = 1$ is trivial since, by definition of link-homotopy, every 1-component string link is homotopically trivial.

If $n > 2$, notice that if we remove the last string we obtain a $(n-1)$-components string link which, using induction hypothesis, we can assume to satisfy the monotonicity condition. If also the last string is decreasing (recall that we are orienting braids and string links with boundary conditions from top to bottom) there is nothing to prove. Otherwise, let $\theta_n = (\theta_{n_x}, \theta_{n_y}, \theta_{n_z}) : I \to D \times I$ a smooth parametrization of the n-th string. Starting from the top and following the orientation of the n-th string, find the first minimum with respect to the horizontal, that is, a point

$$x_m \coloneqq \min\{x \in I \mid \theta'_{n_z}(x) = 0 \text{ and } \theta''_{n_z}(x) > 0\}.$$

Since the final endpoint of the string is at level 0, there must be a point

$$x_M \coloneqq \min\{x \in I \mid x > x_m,\ \theta'_{n_z}(x) = 0 \text{ and } \theta''_{n_z}(x) < 0\}.$$

Observe that x_m and x_M exist since they are the minima over non-empty, finite sets of real numbers. So, x_m and x_M are respectively the first local minimum and maximum of the last string. If the piece of curve $\theta_n([x_m, x_M])$ doesn't loop around the other components of the string link, then via a homotopy (self-intersections are allowed) we can get rid of this pair of critical points. If, instead, the portion of string included between $\theta_n(x_m)$ and $\theta_n(x_M)$ loops around some of the others components we proceed as follows. Since the first $n-1$ strings form a pure braid, we transform it in normal form. Up to isotopy, by sliding down the last component along the others as shown with an example in Figures 3.1(a)-3.1(b), we can assume that $\theta_n([x_m, 1])$ is contained in a small region of $D \times I$ where the first $n-1$ strings are straight. At this point, we can apply an homotopy to make θ_n downward monotonous and write the n-th string as a product of generators powers

$$A_{n,i_1}^{j_1} A_{n,i_1}^{j_1} \dots A_{n,i_k}^{j_k}$$

where $k \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{Z}$ and $i_1, \dots, i_n \in \{1, \dots, n-1\}$ are the indexes (eventually with repetitions) of the components involved in crossings with the last strand.

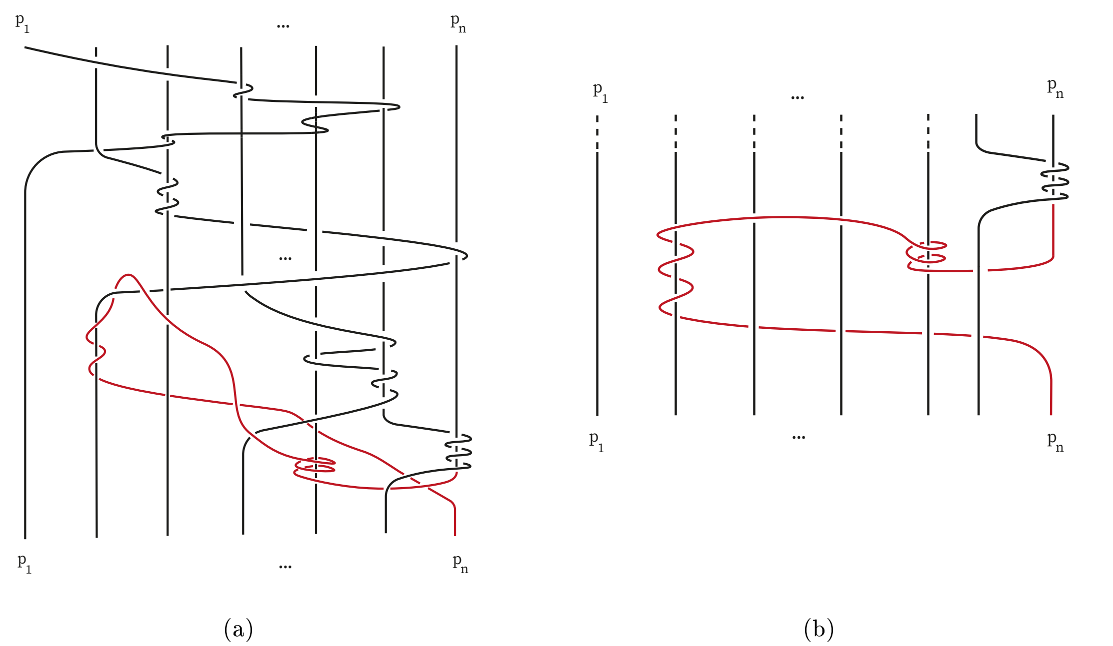


Figure 3.1: A simple example of sliding down the last string.

Notice that the resulting braid is related to the projection of the n-th string on the unit disk seen as an element of $\pi_1(D \setminus \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{n-1}\})$.
For the general case, let now θ be an n-string link and π_θ its associated permutation, that is, the i-th string starts from $(p_i, 1)$ and terminates at $(p_{\pi_\theta(i)}, 0)$ for each $i = 1, \dots, n$. Choose an n-braid β such that $\pi_\beta = \pi_\theta$ and compose θ with the trivial braid $\beta^{-1}\beta$ in $D \times I$. Clearly, we obtain a string link isotopically equivalent to the original one. The composition of θ and

β^{-1} is a pure string link, for which we can apply the procedure described above, turning it through homotopy into a pure braid β'. Then, $\beta'\beta$ is link-homotopic to θ and it represents the desired braid. □

Thus, when studying braids up to link-homotopy we can consider them as string links and so we can ignore the monotonic condition. This fact is quite useful for our purposes, since clasper surgery does not respect this condition in general.

We introduce now *comb-claspers* and their associated notation. Let 1_n be the usual representative of the trivial n-braid ($1_n = \bigsqcup_i \{p_i\} \times I$). Denote by $(D \times I)^+$ and $(D \times I)^-$ the two half-cylinders determined by the xz plane. We choose $(D \times I)^+$ to be above the plane of projection.

Definition 3.2. We call *comb-clasper* a simple clasper without repeats for the trivial braid such that:

- Every edge is in $(D \times I)^+$;
- the minimal path running (and oriented) from the smallest (with respect to the components indexes) to the largest component of the support meets all nodes;
- At each node, the edge that is not part of the minimal path leaves "to the left" as locally depicted in Figure 3.2.

Definition 3.2 allows us to order the support of a comb-clasper: we start with the smallest component, then we arrange the components according to the order in which we meet them along the minimal path and, finally, we end with the largest one. For example, in Figure 3.3 the ordered support is given by $\{1, 2, 6, 4, 5, 8\}$.

Once we fixed the ordered support $\{i_1, \ldots, i_m\}$ of a comb-clasper, it remains to determine the embedding of its edges in $(D \times I)^+$. Note that up to link-homotopy the signs of

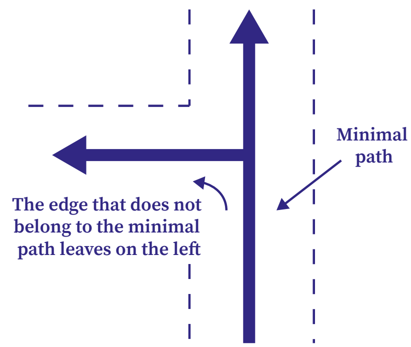


Figure 3.2: Local representation of a node in a comb-clasper

crossings between the edges are irrelevant (move (1) of Corollary 2.22). By Remark 2.25, we can suppose that the comb-claspers contains either one or no twists and, moreover, by Lemma 2.24 we can move the potential twist on the edge connected to the i_m-th component. Therefore, we can unambiguously denote by $(i_1, \ldots, i_m)$ the twisted comb-clasper and by $(i_1, \ldots, i_m)^{-1}$ the untwisted one. If it is not confusing, we will often write the sequence $(i_1 \ldots i_m)$ without commas.

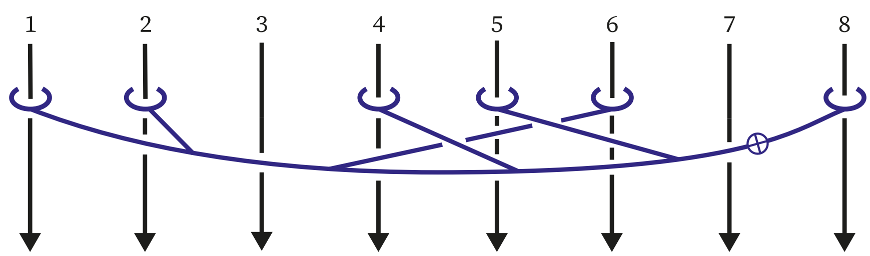


Figure 3.3: The twisted comb-clasper (126458)

Example 3.3. Let $n = 4$. In Figures 3.4(a)-3.4(c) it is shown how we get the braid $\beta = A_{13}^{-1} A_{14} A_{13} A_{14}^{-1} = \left[A_{13}^{-1}, A_{14}\right]$ from surgery on the comb-clasper $(134)^{-1}$.

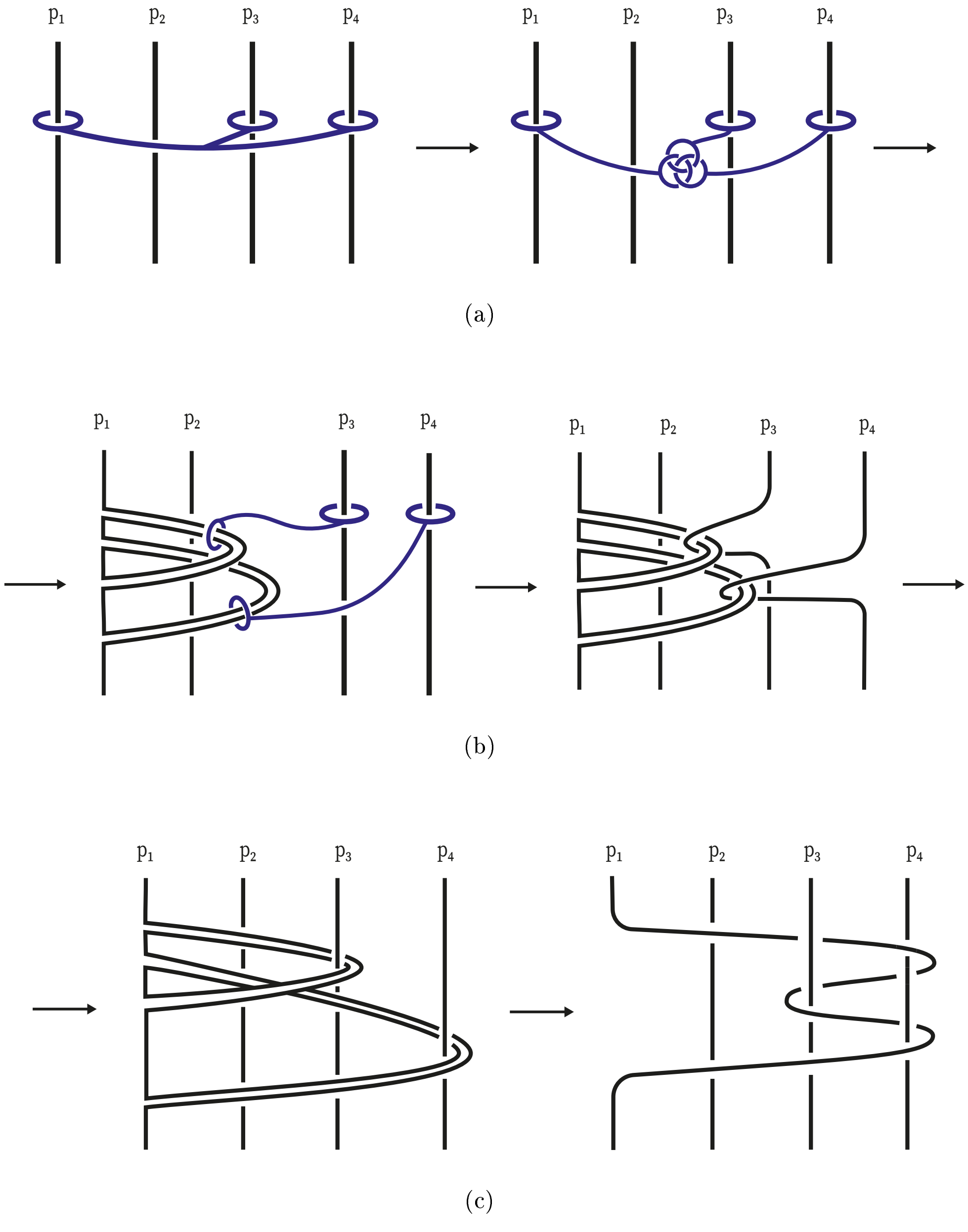


Figure 3.4

From here on, we identify comb-claspers for the trivial n-braid and the result of their surgery up to link-homotopy. In this way, we will consider a comb-clasper as a pure homotopy braid. We define the product of two comb-claspers $(\alpha)(\alpha')$ as the composition of their surgery results $1_n^{(\alpha)}1_n^{(\alpha')}$. Notice that by move (1) from Corollary 2.22, we have $(\alpha)(\alpha)^{-1} = 1_n^{(\alpha)}1_n^{(\alpha)^{-1}} = 1_n$.

Let us see how we can turn any simple clasper for the trivial braid in a product of comb-claspers.

Lemma 3.4. *Let T be a simple clasper of degree k for the trivial braid. Then 1^T is link-homotopic to a product of comb-claspers with degree greater or equal to k.*

Proof. First, by move (4) of Corollary 2.22 and ambient isotopies we can turn T into a product of claspers whose edges are all in $(D \times I)^+$, perhaps creating claspers of higher degree (corresponding to $\tilde{T}$ in move (4)). By the IHX relation we can assume that for each factor the minimal path running from the small to the largest component meets all its nodes. In fact, if there is a node of the clasper that does not belong to this minimal path, by IHX relation we can decompose it into the union of two claspers that suit our needs, as depicted in Figure 3.5 (for more details, see [13, Lemma 2.11]). Finally, by move (8) of Lemma 2.24 we can make sure that the third condition of Definition 3.2 is also satisfied and we obtain a product of comb-claspers. □

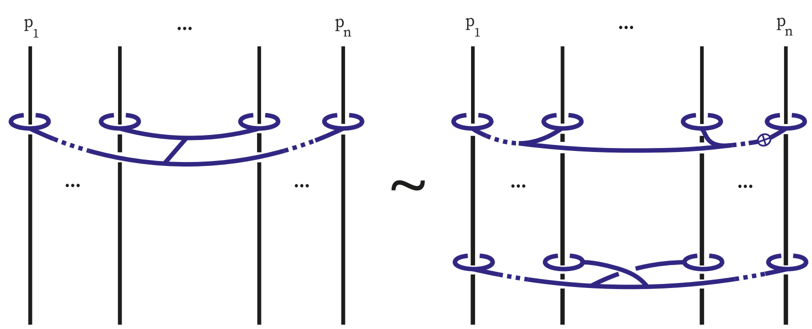


Figure 3.5: How to use IHX relation to turn a clasper into a product of comb claspers

Definition 3.5. We say that a pure homotopy braid $\beta = (\alpha_1)^{\pm 1}(\alpha_2)^{\pm 1}\ldots(\alpha_m)^{\pm 1} \in \tilde{P}_n$

given by a product of comb-claspers is:

- *stacked* if $(\alpha_i) = (\alpha_j)$ for some $i \leq j$ implies that $(\alpha_i) = (\alpha_k)$ for any $i \leq k \leq j$;
- *reduced* if any two consecutive factors are not the inverse of each other.

If β is reduced and stacked we can write

$$\beta = \prod_i (\alpha_i)^{\nu_i},$$

where $i \in \mathbb{N}$ is such that α_i is in the set of comb-claspers with $\text{supp}(\alpha_i) \subset \{1, \ldots, n\}$, $\nu_i \in \mathbb{Z}$ (possibly null) and $(\alpha_i) \neq (\alpha_j)$ for any $i \neq j$.

Definition 3.6. Given an order $\leq$ on the set of twisted comb-claspers, we say that a reduced and stacked product

$$\beta = \prod_i (\alpha_i)^{\nu_i}$$

is a *normal form* of β for this order if $(\alpha_i) \leq (\alpha_j)$ for all $i \leq j$.

Notice that the notion of normal form clearly depends on the chosen order on the set of twisted comb-claspers.

Example 3.7. Given two twisted comb-claspers $(\alpha) = (i_1 \ldots i_l)$ and $(\alpha') = (i'_1 \ldots i'_l)$ we can choose the order $(\alpha) \leq (\alpha')$ defined by:

- $\max(\text{supp}(\alpha)) \leq \max(\text{supp}(\alpha'))$, or
- $\max(\text{supp}(\alpha)) = \max(\text{supp}(\alpha'))$ and $\deg(\alpha) < \deg(\alpha')$, or
- $\max(\text{supp}(\alpha)) = \max(\text{supp}(\alpha'))$ and $\deg(\alpha) = \deg(\alpha')$ and $(i_1, \ldots, i_l) <_{lex} (i'_1, \ldots i'_l)$,

where $<_{lex}$ is the lexicographic order.

Take for example $n = 4$. With respect to this order the normal form of an element $\beta \in \tilde{P}_n$

is given by the integers $\{\nu_{12}, \ldots, \nu_{1324}\}$ as follows:

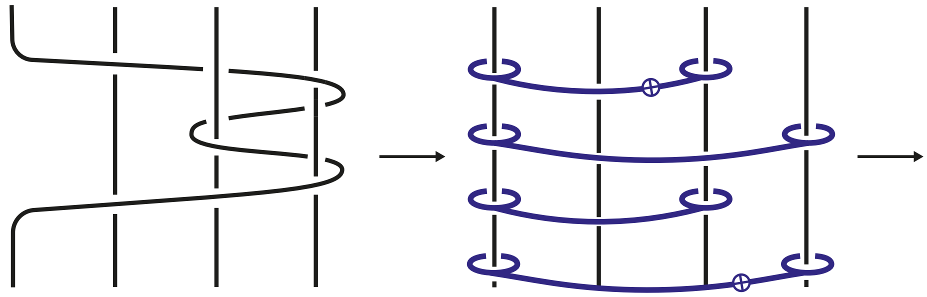
$$\beta = (12)^{\nu_{12}}(13)^{\nu_{13}}(23)^{\nu_{23}}(123)^{\nu_{123}}(14)^{\nu_{14}}(24)^{\nu_{24}}(34)^{\nu_{34}}(124)^{\nu_{124}} \\ (134)^{\nu_{134}}(234)^{\nu_{234}}(1234)^{\nu_{1234}}(1324)^{\nu_{1324}}.$$

Another important result about the description of braids in terms of claspers is given by [6, Theorem 3.7].

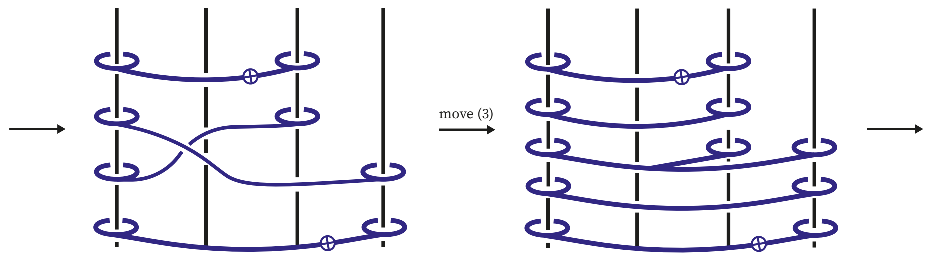
Theorem 3.8. *Any pure homotopy braid $\beta \in \tilde{P}_n$ can be expressed in normal form for any chosen order on the set of twisted comb-claspers.*

Proof. Note that the comb-clasper (ij) corresponds to the Artin pure braid group generator $A_{ij} \in \tilde{P}_n$ (recall Remark 2.3). Thus, it is clear that $\beta = \prod_k (\alpha_k)^{\pm 1}$ for some degree one comb-claspers $(\alpha_k)^{\pm 1}$ and with possible repetitions in the product. Rearrange this product according to the order on the set of comb-claspers, using moves (3) and (5) from Corollary 2.22. This step may introduce new claspers of degree strictly higher than one: by Lemma 3.4 we can assume that these are all comb-claspers. Among these new comb-claspers, consider those of degree two and rearrange them according to the chosen order; again this introduces higher degree factors, which can be assumed to be comb-claspers. By iterating this procedure degree by degree we obtain the desired normal form. Indeed, the procedure terminates after a finite number of steps, since by Lemma 2.21 claspers of degree greater than n are trivial in $\tilde{P}_n$. □

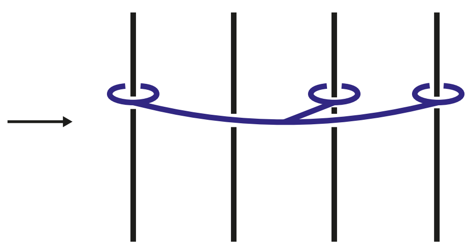
Example 3.9. Consider again $\beta = \left[A_{13}^{-1}, A_{14}\right] \in P_4$. In Figures 3.6(a)-3.6(c) it is shown the procedure to express β as a product of comb-claspers in normal form. In particular, in Figure 3.6(b) we use move (3) from Corollary 2.22 and after simplifying products of claspers with their inverses we get $\beta = (134)^{-1}$, as we wanted.



(a)



(b)



(c)

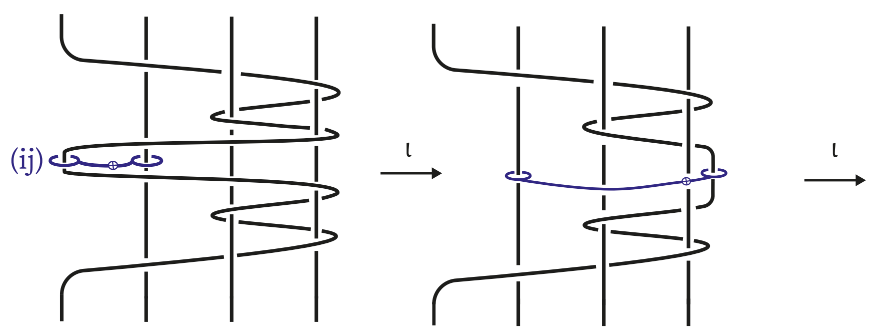
Figure 3.6: The final normal form for $\beta = \left[A_{13}, A_{14}^{-1}\right] = (134)^{-1}$

We conclude with another presentation of the pure homotopy braid group introduced by Graff in [6], which differs from Artin's homotopic representation (see Section 1.3) and where clasper calculus is used.

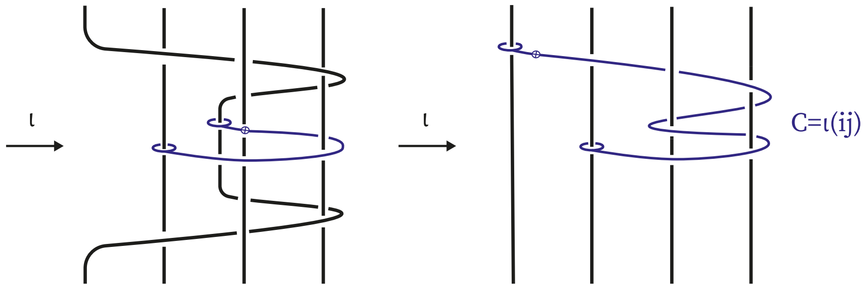
Theorem 3.10. *Let $J' \lhd B_n$ denote the normal subgroup generated by all elements of the form $[A_{ij}, \lambda A_{ij}\lambda^{-1}]$ with $\lambda \in P_n$. Then we have:*

$$\tilde{B}_n = {}^{B_n}\big/_{J'}.$$

Proof. As we have seen, in [5] $\tilde{B}_n$ appears as B_n/J where $J \lhd B_n$ is the normal subgroup generated by elements of the form $[A_{ij}, \lambda A_{ij}\lambda^{-1}]$, where λ belongs to the normal subgroup generated by $\{A_{1,j}, \dots, A_{j-1,j}\}$. The thesis follows from the fact that $J' = J$. Indeed we clearly have $J \subset J'$ since $A_{ij} \in P_n$. We need to show that $J' \subset J$. For this, we prove that for any $\Lambda \in P_n$, A_{ij} and $\Lambda A_{ij}\Lambda^{-1}$ commute up to link homotopy, i.e. $[A_{ij}, \Lambda A_{ij}\Lambda^{-1}] = 1_n \in \tilde{B}_n$. We remind that A_{ij} is the surgery result $1_n^{(ij)}$ of the comb-clasper (ij). Thus, the conjugate $\Lambda A_{ij}\Lambda^{-1}$ is the surgery result of the clasper $C = \iota(ij)$ where ι is the ambient isotopy sending $\Lambda\Lambda^{-1}$ to 1_n (see for example Figures 3.7(a) and 3.7(b)). Being ι an isotopy, $\mathrm{supp}(C) = \mathrm{supp}(ij)$ and hence by Remark 2.23 (here $|\mathrm{supp}(C) \cap \mathrm{supp}(ij)| = 2$) we have $(ij)C \sim C(ij)$. □



(a)



(b)

Figure 3.7: An isotopy from $\Lambda\Lambda^{-1}$ to 1_4 with $\Lambda = \left[A_{13}^{-1}, A_{14}\right]$. The basic comb-clasper (ij) is sent to $C = \iota(ij)$ (here $(i,j) = (1,2)$).

A similar result holds for the pure homotopy braid group $\tilde{P}_n$. In order to prove it, we need the following.

Lemma 3.11. *The subgroup $J \lhd B_n$ normally generated in B_n by elements of the form $[A_{ij}, \lambda A_{ij}\lambda^{-1}]$ for $\lambda \in P_n$, seen as a subgroup of P_n, coincide with the normal subgroup of P_n generated by elements of the form $[A_{ij}, \lambda A_{ij}\lambda^{-1}]$ with $\lambda \in P_n$*

Proof. Let $k \in \{1, \ldots, n-1\}$, $1 \le i < j \le n$ and $\lambda \in P_n$. Then we have:

$$\sigma_k[A_{ij}, \lambda A_{ij}\lambda^{-1}]\sigma_k^{-1} = \begin{cases} [A_{i+1j}, \lambda_1 A_{i+1j}\lambda_1^{-1}] & \text{if } i = k \text{ and } j \neq k+1 \\ [A_{ij+1}, \lambda_2 A_{ij+1}\lambda_2^{-1}] & \text{if } j = k \\ A_{k,k+1}[A_{i-1,j}, \lambda_3 A_{i-1,j}\lambda_3^{-1}] & \text{if } i = k+1 \\ A_{k,k+1}[A_{i,j-1}, \lambda_4 A_{i,j-1}\lambda_4^{-1}] & \text{if } i \neq k \text{ and } j = k+1 \\ [A_{ij}, \lambda A_{ij}\lambda^{-1}] & \text{otherwise} \end{cases}$$

for some $\lambda_i \in P_n$, $i \in \{1, 2, 3, 4\}$. In fact, recall from 1.26 the explicit expression of the pure braid generators A_{ij} in terms of the Artin's generators σ_i. Then in the case $i = k$,

$j \neq k+1$, we have that $|k-j| > 1$ and we can write:

$$\begin{aligned}
\sigma_k A_{kj} &= \sigma_k \sigma_{j-1} \dots \sigma_{k+2} \sigma_{k+1} \sigma_k^2 \sigma_{k+1}^{-1} \sigma_{k+2}^{-1} \dots \sigma_{j-1}^{-1} = \\
&= \sigma_{j-1} \dots \sigma_{k+2} \sigma_k \sigma_{k+1} \sigma_k \sigma_k \sigma_{k+1}^{-1} \sigma_{k+2}^{-1} \dots \sigma_{j-1}^{-1} = \\
&= \sigma_{j-1} \dots \sigma_{k+2} \sigma_{k+1} \sigma_k \sigma_{k+1} \sigma_k \sigma_{k+1}^{-1} \sigma_{k+2}^{-1} \dots \sigma_{j-1}^{-1} = \\
&= \sigma_{j-1} \dots \sigma_{k+2} \sigma_{k+1} \sigma_{k+1} \sigma_k \sigma_{k+1} \sigma_{k+1}^{-1} \sigma_{k+2}^{-1} \dots \sigma_{j-1}^{-1} = \\
&= \sigma_{j-1} \dots \sigma_{k+2} \sigma_{k+1}^2 \sigma_k \sigma_{k+2}^{-1} \dots \sigma_{j-1}^{-1} = \\
&= \sigma_{j-1} \dots \sigma_{k+2} \sigma_{k+1}^2 \sigma_{k+2}^{-1} \dots \sigma_{j-1}^{-1} \sigma_k = \\
&= A_{k+1,j} \sigma_k.
\end{aligned}$$

Hence $A_{kj} = \sigma_k^{-1} A_{k+1,j} \sigma_k$. If we replace this expression in the conjugate $\sigma_k [A_{ij}, \lambda A_{ij} \lambda^{-1}] \sigma_k^{-1}$ we get:

$$\begin{aligned}
\sigma_k [A_{kj}, \lambda A_{kj} \lambda^{-1}] \sigma_k^{-1} &= \sigma_k (A_{kj} \lambda A_{kj} \lambda^{-1} A_{kj}^{-1} \lambda A_{kj}^{-1} \lambda^{-1}) \sigma_k^{-1} = \\
&= A_{k+1,j} \sigma_k \lambda \sigma_k^{-1} A_{k+1,j} \sigma_k \lambda^{-1} \sigma_k^{-1} A_{k+1,j}^{-1} \sigma_k \lambda \sigma_k^{-1} A_{k+1,j}^{-1} \sigma_k \lambda^{-1} \sigma_k^{-1} = \\
&= \left[A_{k+1,j}, \lambda_1 A_{k+1,j} \lambda_1^{-1} \right],
\end{aligned}$$

where we set $\lambda_1 = \sigma_k \lambda \sigma_k^{-1} \in P_n$.

If $j = k$, we have that $\sigma_k A_{ik} = A_{ik+1} \sigma_k$, so $A_{ik} = \sigma_k^{-1} A_{ik+1} \sigma_k$ and we can write:

$$\begin{aligned}
\sigma_k \left[A_{ik}, \lambda A_{ik} \lambda^{-1} \right] \sigma_k^{-1} &= \sigma_k (A_{ik} \lambda A_{ik} \lambda^{-1} A_{ik}^{-1} \lambda A_{ik}^{-1} \lambda^{-1}) \sigma_k^{-1} = \\
&= A_{ik+1} \sigma_k \lambda \sigma_k^{-1} A_{ik+1} \sigma_k \lambda^{-1} \sigma_k^{-1} A_{ik+1}^{-1} \sigma_k \lambda \sigma_k^{-1} A_{ik+1}^{-1} \sigma_k \lambda^{-1} \sigma_k^{-1} = \\
&= A_{ik+1} \lambda_2 A_{ik+1} \lambda_2^{-1} A_{ik+1}^{-1} \lambda_2 A_{ik+1}^{-1} \lambda_2^{-1} = \\
&= \left[A_{ik+1}, \lambda_2 A_{ik+1} \lambda_2^{-1} \right]
\end{aligned}$$

where we set $\lambda_2 = \sigma_k \lambda \sigma_k^{-1}$. The other equalities can be proved similarly. Therefore the conjugates $\sigma_k [A_{ij}, \lambda A_{ij} \lambda^{-1}] \sigma_k^{-1}$ are always conjugates of $[A_{i',j'}, \lambda' A_{i'j'} (\lambda')^{-1}]$ for suitable $i', j' \in \{1, \dots, n\}$ and $\lambda' \in P_n$. $\square$

From [6, Corollary 3.22] we have the following.

Corollary 3.12. *Let $J \lhd P_n$ be the normal subgroup generated by elements of the form $[A_{ij}, \lambda A_{ij} \lambda^{-1}]$ for any $\lambda \in P_n$. Then we obtain the pure homotopy braid group as:*

$$\tilde{P}_n = {}^{P_n}\!/_{J} = RP_n.$$

This induces the following presentation for $\tilde{P}_n$:

$$\tilde{P}_n = \left\langle A_{ij} \;\middle|\; \begin{array}{ll} [A_{r,s}, A_{i,j}] = 1 & r < s < i < j \text{ or } r < i < j < s \\ [A_{r,s}, A_{r,j}] = [A_{r,j}, A_{s,j}] = [A_{s,j}, A_{r,s}] & r < s < j \\ [A_{r,i}, A_{s,j}] = [[A_{i,j}, A_{r,j}], A_{s,j}] & r < s < i < j \\ [A_{ij}, \lambda A_{ij} \lambda^{-1}] = 1 & i < j \text{ and } \lambda \in P_n \end{array} \right\rangle.$$

Proof. The first part of the statement is a direct consequence of Theorem 3.10 and Lemma 3.11. For the presentation of $\tilde{P}_n$, Graff obtains it from that of [18, Theorem 3.8] re-expressed in terms of commutator and using the relation $[A_{r,s}, A_{i,j}^{-1}] = [A_{r,s}, A_{i,j}]^{-1}$, which holds in $\tilde{P}_n$. $\square$

3.2 Algebraic counterpart

In this section we give the algebraic counterpart of the comb-claspers: the *reduced basic commutators.* The notion of basic commutators was first introduced in [10] and then adapted to the framework of the reduced free group in [14]; although, we present here a different choice of the set of basic commutators, again following [6].

Recall from Section 1.2 that F_n denotes the free group on n generators and RF_n is its reduced free group.

Definition 3.13. A *commutator* in $\{x_1, \ldots, x_n\}$ of F_n is defined recursively as follows:

- the commutators of weight one are $x_1, \ldots, x_n$;
- $C = [C_1, C_2]$ is a commutator of weight $k > 1$ if $\mathrm{wg}(C) := \mathrm{wg}(C_1) + \mathrm{wg}(C_2) = k$.

Definition 3.14. We define the *number of occurances* of the generator x_i in a commutator C, denoted by $\mathrm{Occ}_i(C)$, as follows:

- $\mathrm{Occ}_i(x_j) = \delta_{ij}$,
- $\mathrm{Occ}_i([C_1, C_2]) = \mathrm{Occ}_i(C_1) + \mathrm{Occ}_i(C_2)$.

We say that a commutator $C \in F_n$ *has repeats* if $\mathrm{Occ}_i(C) > 1$ for some $i = 1, \ldots, n$, while the *support* of C, $\mathrm{supp}(C)$, is the set of indices i such that $\mathrm{Occ}_i(C) > 0$.

From Lemma 2.21 we know that all comb-claspers with repeats are trivial up to link-homotopy. An analogous result holds for commutators with repeats.

Proposition 3.15. *The subgroup J_{F_n} is generated by commutators in $\{x_1, \ldots, x_n\}$ with repeats. Hence, commutators with repeats are trivial in the reduced free group.*

Proof. See [14, Proposition 3]. □

Definition 3.16. We call *set of reduced basic commutators* the family $\mathcal{F}$ of commutators without repeats in RF_n defined as:

$$\mathcal{F} = \{[i_1, \cdots, i_l] \mid i_1 < i_k,\ 2 \leq l\}_{l \leq n}$$

where we use the notation $[i_1, \cdots, i_l] := [[\cdots[[x_{i_1}, x_{i_2}], x_{i_3}], \cdots, x_{i_{l-1}}], x_{i_l}]$.

If there is no risk of confusion, we will omit the commas inside the square brackets of the commutators $[i_1 \cdots i_l]$ to simplify the notation.

Example 3.17. Let $[\alpha] = [i_1, \cdots, i_l]$ and $[\alpha'] = \left[i'_1, \cdots, i'_{l'}\right]$. We can choose the order $\alpha \leq \alpha'$ given by:

- $\mathrm{wg}(\alpha) < \mathrm{wg}(\alpha')$, or
- $\mathrm{wg}(\alpha) = \mathrm{wg}(\alpha')$ and $(i_1, \cdots, i_l) <_{lex} (i'_1, \ldots, i'_l)$.

Observe that if $\mathrm{wg}(\alpha) = \mathrm{wg}(\alpha')$ then l and l' must coincide. Take for example $n = 3$. In this situation, the normal form of an element $\omega \in RF_3 = \langle x_1, x_2, x_3 \rangle$ is given by:

$$\omega = [1]^{e_1}[2]^{e_2}[3]^{e_3}[12]^{e_{12}}[13]^{e_{13}}[23]^{e_{23}}[123]^{e_{123}}[132]^{e_{132}}, \quad e_i \in \mathbb{Z}.$$

The following theorem states the existence and uniqueness of a normal form for every element in RF_n. It involves advanced algebraic tools, thus we will not include the proof, which can be found in [6, Theorem 3.15].

Theorem 3.18. *For any word $\omega \in RF_n$ there exists a unique set of integers $\{e_1, \ldots, e_m\}$ associated to the set $\mathcal{F} = \{[\alpha_1], [\alpha_2], \ldots, [\alpha_m]\}$, ordered and indexed in an arbitrary way, such that $\omega = [\alpha_1]^{e_1}[\alpha_2]^{e_2} \ldots [\alpha_m]^{e_m}$.*

We can proceed introducing some other new ingredients which we will need in order to define a linear representation of the homotopy braid group. Given the set of reduced basic commutators $\mathcal{F} = \{[\alpha_1], [\alpha_2], \ldots, [\alpha_m]\}$, we associate to it the $\mathbb{Z}$-module $\mathcal{V}$ formally generated by $\{\alpha_1, \ldots, \alpha_m\}$.

Definition 3.19. We define the *linearization map* ϕ as

$$\begin{aligned}\phi : RF_n &\to \mathcal{V}\\ \omega &\mapsto \phi(\omega) = e_1\alpha_1 + \cdots + e_m\alpha_m\end{aligned}$$

where $[\alpha_1]^{e_1}[\alpha_2]^{e_2}\dots[\alpha_m]^{e_m}$ is the normal form of ω.

We keep calling "commutators" the generators of $\mathcal{V}$ and we define the weight and support of α to be the ones of $[\alpha]$. Notice that the normal form (and hence the linearization maps, too) depends on the ordering of $\mathcal{F}$.

Lemma 3.20. *1. The $\mathbb{Z}$-module $\mathcal{V}$ has rank*

$$rk(\mathcal{V}) = \sum_{0\le l\le k<n} \frac{k!}{l!};$$

2. $\mathcal{V}$ can be decomposed into a direct sum of submodules $\mathcal{V}_i$ generated by commutators of weight i and we have

$$rk(\mathcal{V}_i) = \sum_{i-1\le k<n} \frac{k!}{(k-i+1)!}.$$

Proof. The first equality comes from counting the cardinality of $\mathcal{F}$. Initially we count the elements $[\alpha]$ with first term $\alpha_1 = k$. To choose $\alpha_2, \alpha_3, \dots, \alpha_l$ with $0 \le l < n-k$ we only have to respect the condition that $\alpha_1 < \alpha_i$. Thus they can be chosen in $\{k+1, \dots, n\}$ and hence:

$$\mathrm{rk}(\mathcal{V}) = \sum_{k=1}^{n}\sum_{l=0}^{n-k} \frac{(n-k)!}{(n-k-l)!} = \sum_{k=0}^{n-1}\sum_{l=0}^{k} \frac{k!}{(k-l)!} = \sum_{k=0}^{n-1}\sum_{l=0}^{k} \frac{k!}{l!}.$$

where the first equality comes from the substitution $\bar{k} = n-k$ and the second one follows by setting $\bar{l} = \bar{k} - l$. Finally we rename indexes.

For the second equality, we have that the first term $\alpha_1 = k$ must be chosen in $\{1, \dots, n-i+1\}$

and then we freely choose the $i-1$ last numbers $\alpha_2, \ldots, \alpha_i$ in $\{k+1, \ldots, n\}$. We obtain:

$$\mathrm{rk}(\mathcal{V}_i) = \sum_{k=1}^{n-i+1} \frac{(n-k)!}{(n-k-i+1)!} = \sum_{k=i-1}^{n-1} \frac{k!}{(k-i+1)!}.$$

where we replaced $n-k$ with k. □

3.3 A linear faithful representation of the homotopy braid group

In this section we give a linear representation of the homotopy braids group and we prove its injectivity using clasper calculus. In order to define this representation $\gamma : \tilde{B}_n \to GL(\mathcal{V})$, we state the following preparatory lemma. Recall that the representation $\tilde{\rho}$ defined in 1.29 is faithful. Denote by N_j the subgroup normally generated by x_j in RF_n for $j \in \{1, \ldots, n\}$; then N_j is abelian and we have:

Lemma 3.21. *Let $\beta \in \tilde{B}_n$. For any commutator $C \in N_j$, if $[\alpha_1]^{e_1} \cdots [\alpha_m]^{e_m}$ is a normal form of $\tilde{\rho}(\beta)(C)$, then we have that $e_i = 0$ if $[\alpha_i] \notin N_{\pi_{\beta^{-1}}(j)}$, where $\pi_{\beta^{-1}}(j)$ is the image of j under the permutation induced by β^{-1}.*

Proof. See [6, Lemma 3.24]. □

In other words, for $C \in N_j$, $x_{\pi_{\beta^{-1}}(j)}$ occurs in each factor of the normal form of $\tilde{\rho}(\beta)(C)$. Recall from Definition 3.19 the linearization map $\phi : RF_n \to \mathcal{V}$.

Theorem 3.22. *The map*

$$\gamma : \tilde{B}_n \to GL(\mathcal{V}),$$

defined by

$$\gamma(\beta)(\alpha) := \phi \circ \tilde{\rho}(\beta)([\alpha]) \tag{3.23}$$

for any $\beta \in \tilde{B}_n$ and $[\alpha] \in \mathcal{F}$, is a well defined homomorphism. Moreover γ does not depend on the chosen order on $\mathcal{F}$.

Remark 3.24. *ϕ is not a homomorphism in general, so it is not clear that γ is a representation.*

Proof. Let β, β' be two homotopy braids, $[\alpha]$ a reduced basic commutator and α its corresponding commutator in $\mathcal{V}$. We choose some $j \in \operatorname{supp}([\alpha])$ so that $[\alpha] \in N_j$. Set $\gamma(\beta')(\alpha) = \sum_i \alpha_i$ for some commutators $\alpha_i \in \mathcal{V}$ associated to the reduced basic commutators $[\alpha_i]$. Then we have that

$$\gamma(\beta\beta') = \phi \circ \tilde{\rho}(\beta)\tilde{\rho}(\beta')([\alpha]) = \phi \circ \tilde{\rho}(\beta)(\prod_i [\alpha_i]) = \phi(\prod_i \tilde{\rho}(\beta)([\alpha_i])),$$

where in the second equality the exponents are all 1 since α is a reduced basic commutator and the last equality follows from the fact that $\tilde{\rho}$ is a homomorphism.
Now, using Lemma 3.21, we know that $[\alpha_i] \in N_{\pi_{\beta'^{-1}}(j)}$ for any i. Furthermore, Lemma 3.21 implies that any commutator in the normal form of $\tilde{\rho}(\beta)([\alpha_i])$ is in the abelian group $N_{\pi_{(\beta\beta')^{-1}}(j)}$ for any i. Note that for $C_1, \ldots, C_k$ a collection of commutators in $\mathcal{F}$ such that $[C_i, C_j] = 1$ for any i, j we have that

$$\phi(C_1 \cdots C_k) = \phi(C_1) + \cdots + \phi(C_k).$$

Hence ϕ behaves like a homeomorphism on the product $\prod_i \tilde{\rho}(\beta([\alpha_i])$ and, finally,

$$\phi\left(\prod_i \tilde{\rho}(\beta)([\alpha_i])\right) = \sum_i \phi\left(\tilde{\rho}(\beta)([\alpha_i])\right) = \sum_i \gamma(\beta)(\alpha_i) = \gamma(\beta)\left(\sum_i (\alpha_i)\right) = \gamma(\beta)\gamma(\beta')(\alpha),$$

so γ is a well defined homomorphism.

To prove the independence on the order on $\mathcal{F}$ we use Lemma 3.21 again. For any

$\beta \in \tilde{B}_n$ and $[\alpha] \in \mathcal{F}$, all the commutators in the normal form of $\tilde{\rho}([\alpha])$ commute with each other (indeed, $[\alpha_i] \in N_{\pi_{\beta^{-1}}(j)}$, which is abelian). In particular, if we set two orderings $\{[\alpha_1], \ldots, [\alpha_m]\}$ and $\{[\alpha_{\sigma(1)}], \ldots, [\alpha_{\sigma(m)}]\}$ on $\mathcal{F}$ then the two associated normal forms

$$\tilde{\rho}([\alpha]) = [\alpha_1]^{e_1} \cdots [\alpha_m]^{e_m} = [\alpha_{\sigma(1)}]^{e'_{\sigma(1)}} \cdots [\alpha_{\sigma(m)}]^{e'_{\sigma(m)}}$$

satisfy $e_i = e'_i$ for any i and therefore $\phi \circ \tilde{\rho} = \phi' \circ \tilde{\rho}$ for the two linearization maps ϕ and ϕ' associated to the two different orderings. □

At this point, the injectivity of the homomorphism γ can be shown using the injectivity of ϕ (it is clearly injective) and $\tilde{\rho}$ (proved in [7]). However, we will give another proof of this result using clasper calculus. This approach allows also an explicit computation of this representation.

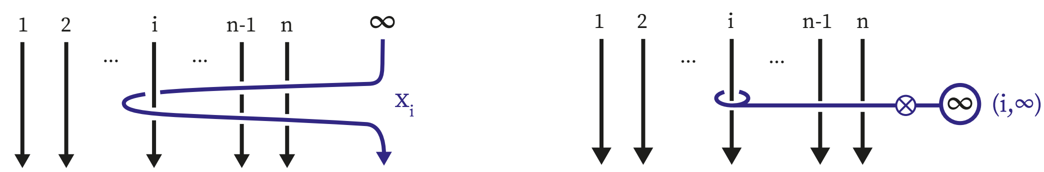


Figure 3.8: Pure braid and clasper interpretation of the generator $x_i \in F_n$.

Recall from [7] that we can see the free group F_n as the fundamental group of the complement of the n-component trivial braid. Similarly, if we consider braids up to homotopy, we have that RF_n acts as the reduced fundamental group of the complement of the trivial n-braid. Therefore, we can identify any element of F_n (or RF_n) with the homotopy class (*reduced homotopy* class) of an $(n+1)$-component braid in this complement, where by reduced homotopy class we mean the image in the reduced quotient of the homotopy class of an element. In the diagram, we place the new strand to the right of the braid and we label it by "∞". In this way, the generators x_i of F_n (RF_n) are given by the pure braids $A_{i,\infty}$, shown in Figure 3.8, which can be reinterpreted with the comb-clasper (i, ∞), drawn in the same figure. From now on, we will represent the leaf intersecting the ∞-th component with a circled "∞".

In this context, the image $\rho(\beta)(x_i) \in F_n$ $(\tilde{\rho}(\beta)(x_i) \in RF_n)$, for $\beta \in B_n$ $(\beta \in \tilde{B}_n)$ and x_i a generator of F_n (RF_n), is given by considering the conjugation $\beta 1_n^{(i,\infty)} \beta^{-1}$ illustrated in Figure 3.9. Then, via an isotopy, we transform $\beta 1_n^{(i,\infty)} \beta^{-1}$ into 1_n. During this process the comb-clasper (i, ∞) is deformed into a new clasper which we are able to reinterpret (after reducing it into a new product of comb-claspers) as an element of F_n (RF_n). More precisely, in the link-homotopic case we have a correspondence between the family $\mathcal{F}$ and the comb-claspers with ∞ in their support.

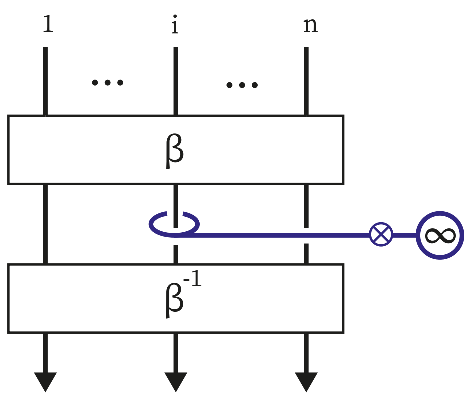


Figure 3.9: The conjugation $\beta 1_n^{(i,\infty)} \beta^{-1}$.

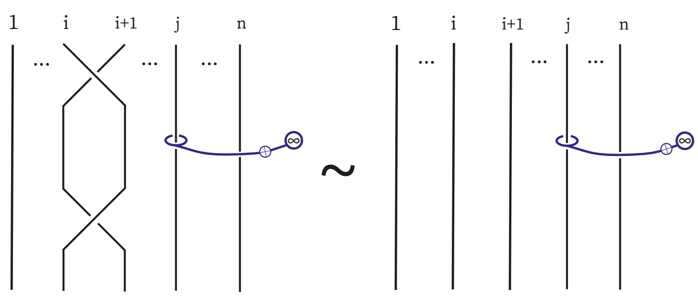
Proposition 3.25. *Let* $(\alpha) = (i_1 \dots i_{n-1} i_n \infty)$ *and* $(\alpha') = (i_1 \dots i_{n-1}, \infty)$ *be two comb-claspers. Then* $(\alpha) \sim [(\alpha'), (i_n \infty)]$.

Proof. Consider the product of comb-claspers $(\alpha')(i_n\infty)(\alpha')^{-1}(i_n\infty)^{-1}$. First, we use move (3) of Corollary 2.22 to exchange the ∞-th leaves of $(i_n\infty)$ and $(\alpha')^{-1}$. This move creates an extra comb-clasper, which is exactly (α). By Remark 2.23 we can freely move (α) and finish exchanging the edges of $(i_n\infty)$ and $(\alpha')^{-1}$, obtaining the product

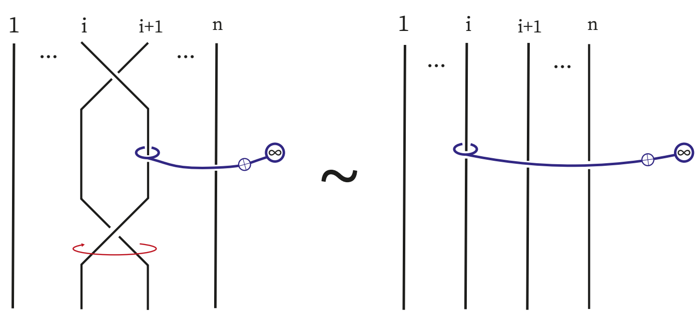
$$(\alpha')(\alpha')^{-1}(\alpha)(i_n\infty)(i_n\infty)^{-1} \sim (\alpha).$$

$\square$

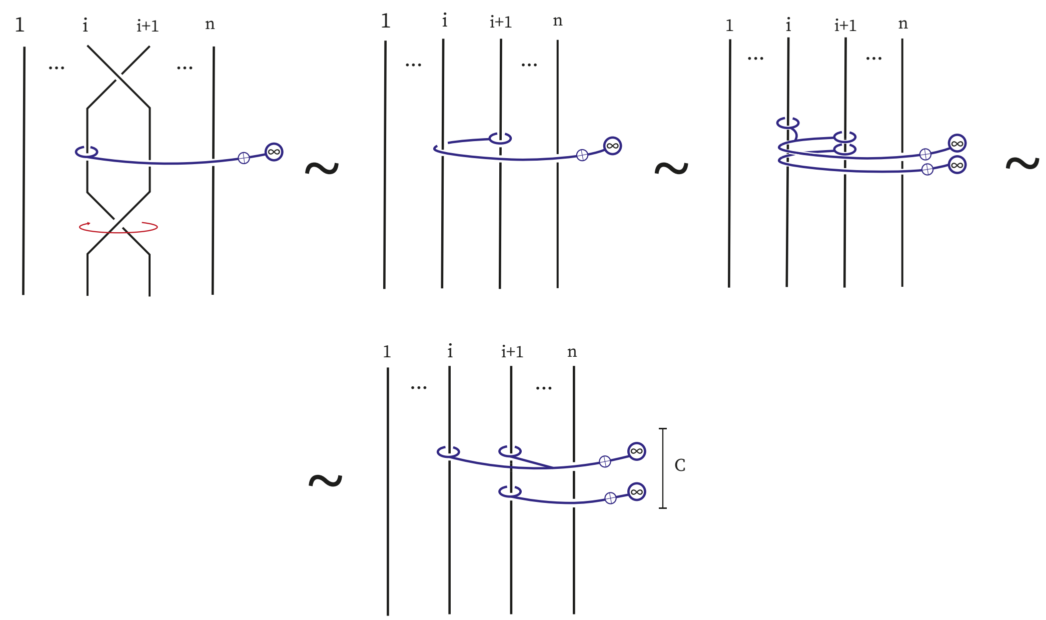
For example, let us compute the image of $x_k \in F_n$ via $\rho(\sigma_i)$ with $i \in \{1, \ldots, n\}$. The case $j \neq i, i+1$ is immediate (see Figure 3.10(a)) and we obtain $\rho(\sigma_i)(x_j) = x_j$. If $j = i+1$, via an isotopy we get $\rho(\sigma_i)(x_{i+1}) = x_i$ as shown in Figure 3.10(b).



(a) Computing $\rho(\sigma_i)(x_j)$, $j \neq i, i+1$



(b) Computing $\rho(\sigma_i)(x_i + 1)$



(c) Computing $\rho(\sigma_i)(x_i)$

Figure 3.10: Computation of the representation ρ on the generators σ_i using clasper calculus.

Finally, if $j = i$ we first apply the same isotopy of the previous case; then by move (4) of Corollary 2.22 we obtain the comb-clasper $C = (i, i+1, \infty)(i+1, \infty)$ (Figure 3.10(c)). By proposition 3.25 we can write:

$$
\begin{aligned}
C = (i, i+1, \infty)(i+1, \infty) &\sim [(i, \infty), (i+1, \infty)]\, (i+1, \infty) = \\
&= (i, \infty)(i+1, \infty)(i, \infty)^{-1}(i+1, \infty)^{-1}(i+1, \infty) = \\
&= (i, \infty)(i+1, \infty)(i, \infty)^{-1},
\end{aligned}
$$

hence $\rho(\sigma_i)(x_i) = x_i x_{i+1} x_i^{-1}$, as described by Equations 1.29.

Iterating Proposition 3.25 we obtain a correspondence between the commutators $[\alpha] \in \mathcal{F}$ and the comb-claspers (α, ∞).

For example, the equivalence

$$(1254\infty) \sim [(125\infty),(4\infty)] \sim [[(12\infty),(5\infty)],(4\infty)] \sim [[[(1\infty),(2\infty)],(5\infty)],(4\infty)]$$

corresponds to $[1254] = [[[x_1,x_2],x_5]\,x_4]$ in RF_n. Now, let us see how the image of a commutator $(i_1,i_2,\ldots,i_l) \coloneqq \phi([i_1,i_2,\ldots,i_l]) \in \mathcal{V}$ by the map $\gamma(\sigma_i)$ depends on the position of the indices i and $i+1$ in the sequence $i_1,i_2,\ldots,i_l$.

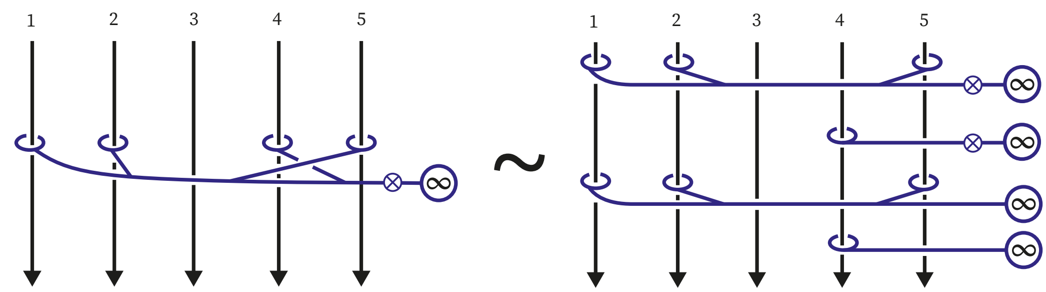


Figure 3.11: The equivalence $(1254\infty) \sim [(125\infty),(4\infty)]$

Theorem 3.26. *For suitable sequences I,J,K in $\{1,\ldots,n\}\setminus\{i,i+1\}$, $I\neq\emptyset$, we have:*

$$\gamma(\sigma_i):\begin{cases}(I) & \mapsto (I) & (a)\\ (J,i,K) & \mapsto (J,i+1,K) & (b)\\ (i+1,K) & \mapsto (i,K)+(i,i+1,K) & (c)\\ (I,i+1,K) & \mapsto (I,i,K)+(I,i,i+1,K)-(I,i+1,i,K) & (d)\\ (I,i,J,i+1,K) & \mapsto (I,i+1,J,i,K) & (e)\\ (I,i+1,J,i,K) & \mapsto (I,i,J,i+1,K) & (f)\\ (i,J,i+1,K) & \mapsto \sum_{J'\subseteq J}(-1)^{|J'|+1}(i,\overline{J'},i+1,J\setminus J',K) & (g)\end{cases}$$

where the sum in (g) is over all (possibly empty) subsequences J' of J, and $\overline{J'}$ denotes the *sequence obtained from J' by reversing the order of its elements.*

Proof. As described above, we consider the conjugate $\sigma_i^{-1}(\alpha,\infty)\sigma_i$ and apply clasper calculus to turn it into a union of comb-claspers.

For (a) it is clear that (I,∞) commutes with σ_i. The computation of (b) is given by an isotopy of the braid shown in Figure 3.12.

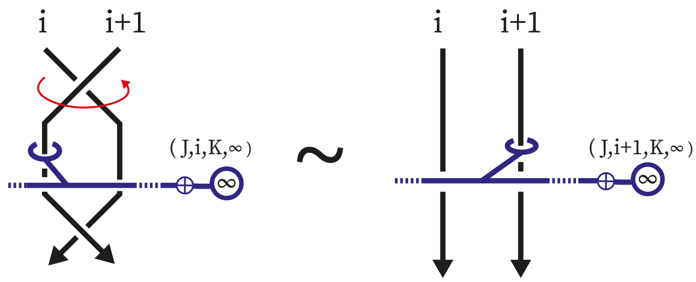


Figure 3.12: Computation of (b).

The proofs of (c) and (d) are similar and are shown in Figures 3.13 and 3.14 respectively. The first equivalence of Figure 3.13 is an isotopy, and the second one is given by move (3) from Corollary 2.22. To prove (d), there is a further step given by an IHX relation.

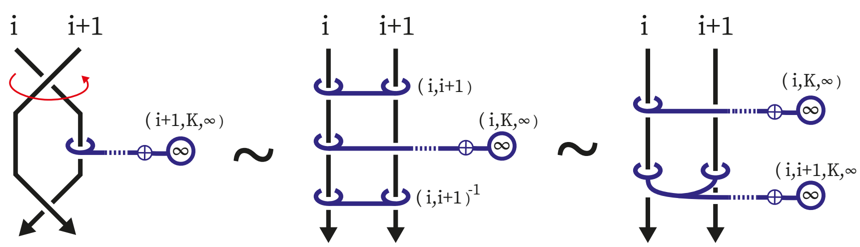


Figure 3.13: Computation of (c).

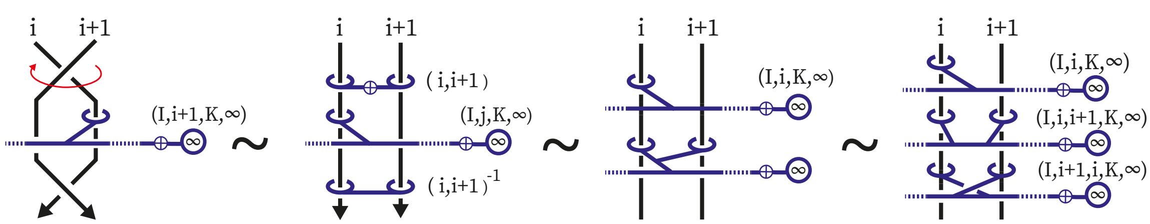


Figure 3.14: Computation of (d).

For (e) and (f) we apply the same isotopy as Figure 3.12 on components i and $i+1$, then we interchange $(I, i, J, i+1, K)$ and $(I, i+1, J, i, K)$. We also need a crossing change between the $(i+1)$-th component and a clasper edge, which is possible according to Remark 2.23.

We prove (g) in two steps. The first step is illustrated in Figure 3.15: as before, the first equivalence is given by an isotopy and a crossing change. Move (8) of Lemma 2.24 ensures us the second one. This procedure turns $\sigma_i(i, J, i+1, K, \infty)\sigma_i^{-1}$ into a new clasper which is not a comb-clasper.

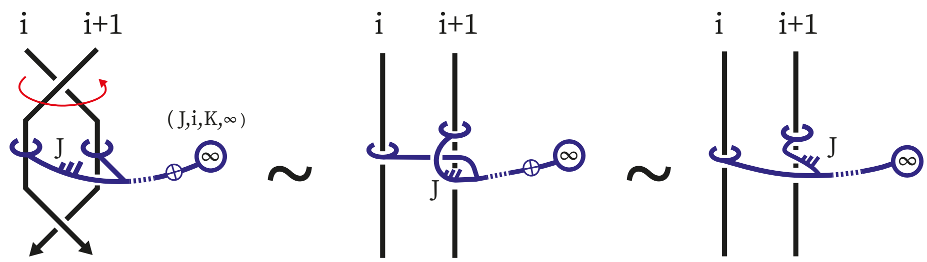


Figure 3.15: Turning $\sigma_i(i, J, i+1, K, \infty)\sigma_i^{-1}$ into a new clasper.

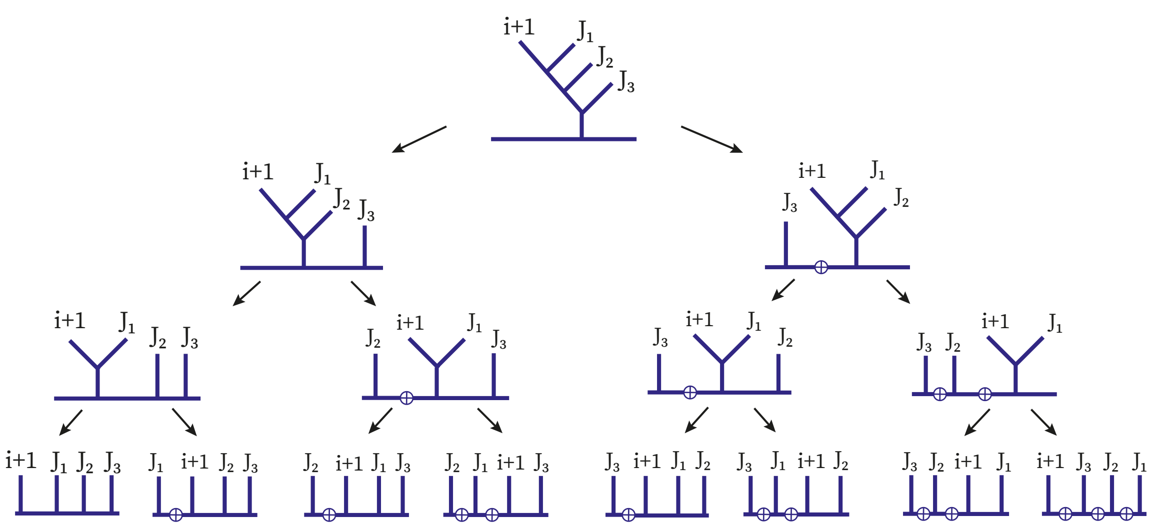


Figure 3.16: The second step of (g).

In the second step we iterate the IHX relations to turn this clasper into a product of comb-claspers. This is illustrated in Figure 3.16, where for instance $J = (j_1, j_2, j_3)$. We conclude by simplifying the twists pairwise (as allowed by Remark 2.23). $\square$

Example 3.27. If $J = (j_1, j_2, j_3)$ and $K = \emptyset$, then

$$\begin{aligned}\gamma(\sigma_i)(i, J, i+1) = &- (i, i+1, j_1, j_2, j_3) + (i, j_1.i+1, j_2, j_3) + (i, j_2, i+1, j_1, j_3) + \\ &+ (i, j_3, i+1, j_1, j_2) - (i, j_2, j_1, i+1, j_3) - (i, j_3, j_1, i+1, j_2) + \\ &- (i, j_3, j_2, i+1, j_1) + (i, j_3, j_2, j_1, i+1).\end{aligned}$$

Example 3.28. We illustrate Theorem 3.26 on the 3-component homotopy braid group $\tilde{B}_3$. Set $(1), (2), (3), (12), (13), (23), (123), (132)$ to be the generators of $\mathcal{V}$, with the order described in Example 3.17 and compute γ on the Artin generators σ_1, σ_2:

$$\begin{aligned}
&\gamma(\sigma_1)(1) = (2) && \gamma(\sigma_2)(1) = (1)\\
&\gamma(\sigma_1)(2) = (1) + (12) && \gamma(\sigma_2)(2) = (3)\\
&\gamma(\sigma_1)(3) = (3) && \gamma(\sigma_2)(3) = (2) + (23)\\
&\gamma(\sigma_1)(12) = -(12) && \gamma(\sigma_2)(12) = (13)\\
&\gamma(\sigma_1)(13) = (23) && \gamma(\sigma_2)(13) = (12) + (123) - (132)\\
&\gamma(\sigma_1)(23) = (13) + (123) && \gamma(\sigma_2)(23) = -(23)\\
&\gamma(\sigma_1)(123) = -(123) && \gamma(\sigma_2)(123) = (132)\\
&\gamma(\sigma_1)(132) = -(123) + (132) && \gamma(\sigma_2)(132) = (132)
\end{aligned}$$

that gives us:

$$\gamma(\sigma_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

In general we have the following.

Proposition 3.29. *Let $\beta \in \tilde{B}_n$ be a homotopy braid. Then, the matrix associated $\gamma(\beta)$ in the basis $\mathcal{F}$, endowed with the order of Example 3.17, is given by a lower triangular block matrix of the form:*

$$\begin{pmatrix} B_{1,1} & 0 & \cdots & 0 \\ B_{2,1} & B_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \cdots & B_{n,n} \end{pmatrix}$$

where $B_{i,i}$ is of order $rk(\mathcal{V}_i) = \sum_{i-1}^{n} \frac{k!}{(k-i+1)!}$, which is the identity when β is pure.

Proof. The triangular shape is a direct consequence of Theorem 3.26; indeed, the chosen order respects the weight, and Theorem 3.26 shows that γ maps a commutator of weight k to a sum of commutators of weight at least k. Lemma 3.20 gives the size of the square diagonal block $B_{i,i}$.

It remains to prove that $B_{i,i}$ is the identity when β is pure. We only need to show this result on the generators $\beta = A_{i,j} = 1^{(i,j)}$. Corollary 2.22 ensures us that conjugating (α, ∞) by (i,j) may only create a clasper (α', ∞) of strictly higher degree. This shows that $\gamma(\beta)(\alpha) = (\alpha) +$ (strictly higher weight commutators), so that $B_{i,i}$ is the identity. □

The following preparatory lemma will allow us to prove the injectivity of γ.

Lemma 3.30. *Let $(i_1, \cdots, i_l)$ be a comb-clasper. We have*

$$\gamma(1^{(i_1,\cdots,i_l)})(i_l) = (i_l) - (i_1, \cdots, i_l),$$

where, on right hand side, $(i_1, \cdots, i_l)$ now denotes the corresponding commutator in $\mathcal{V}$.

Proof. Consider the product $(i_1, \cdots, i_l)(i_d, \infty)(i_1, \cdots, i_l)^{-1}$ and re-write it as a product of only comb-claspers with ∞ in their support. To do so, as illustrated in Figure 3.17, we apply move (3) from Corollary 2.22 on the leaves of the i_d-th component, which introduces the comb-clasper $(i_1, \cdots, i_l, \infty)^{-1}$. Now we simplify $(i_1, \cdots, i_l)$ and $(i_1, \cdots, i_l)^{-1}$. $\square$

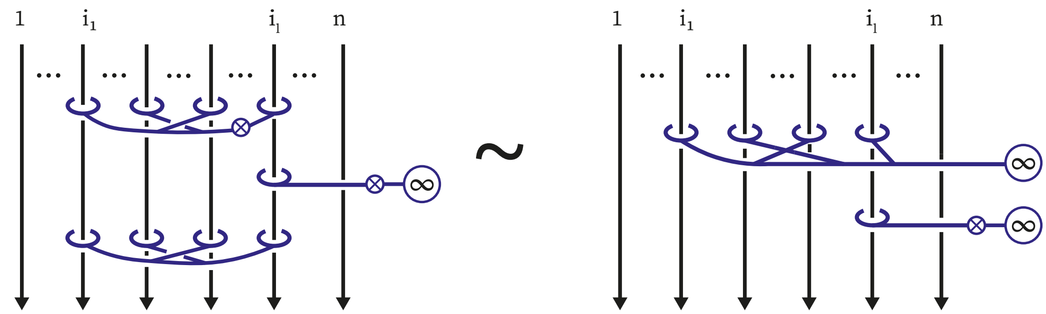


Figure 3.17: Proof of Lemma 3.30

We can now state the main result of this section.

Theorem 3.31. *The representation $\gamma : \tilde{B}_n \to GL(\mathcal{V})$ is injective.*

Proof. Let $\beta \in \tilde{B}_n$ be such that $\gamma(\beta) = \mathrm{id}$. First, Proposition 3.29 imposes that β is a pure braid, since the block $B_{1,1}$ must be the identity, which means that the permutation $\pi(\beta)$ associated to β is trivial.

According to Theorem 3.8, we can consider a normal form for β:

$$\beta = \prod (\alpha)^{\nu_\alpha}.$$

Let $I \subset \{1, \ldots, n\}$ be a sequence of indices with largest index m. Let also $\mathcal{V}_I$ be the subspace of $\mathcal{V}$ spanned by commutators with support included in I. Then we define

$$p_I : \mathcal{V} \to \mathcal{V}_I \quad \text{and} \quad \gamma_I \coloneqq p_I \circ \gamma_{|\mathcal{V}_I},$$

where p_I is the canonical projection with respect to the basis specified above. Using Corollary 2.22 we have that $\gamma(\tilde{P}_n)(\mathcal{V} \setminus \mathcal{V}_I) \subset \mathcal{V} \setminus \mathcal{V}_i$, thus for $\beta_1, \beta_2 \in \tilde{P}_n$ we have that $\gamma_I(\beta_1\beta_2) = \gamma_I(\beta_1)\gamma_I(\beta_2)$. Moreover, $\gamma_I(1^{(\alpha)}) = \mathrm{id}$ for any comb-clasper (α) with $\mathrm{supp}(\alpha) \not\subset I$. Hence, if we set

$$\beta' = \prod_{\mathrm{supp}(\alpha)\subset I} (\alpha)^{\nu_\alpha},$$

we have that $\gamma_I(\beta) = \gamma_I(\beta')$.

Now we show by induction on the degree of (α) that $\nu_\alpha = 0$. For the base case we consider I of the form $I = \{i, m\}$. By Lemma 3.30 we get:

$$\gamma_I(\beta')(m) = \gamma_I(1_n^{(im)^{\nu_{im}}})(m) =$$
$$= (m) - \nu_{im} \cdot (im).$$

Since $\beta \in \ker(\gamma)$, we have that $\gamma_I(\beta)(m) = (m)$, so $\nu_\alpha = 0$ for any (α) of degree one. To prove that $\nu_\alpha = 0$ for any α of degree k, with $k > 0$, we take I of length $k+1$ and, using induction hypothesis, we can write:

$$\beta' = \prod_{\mathrm{supp}(\alpha)\subset I} (\alpha)^{\nu_\alpha} = \prod_{\mathrm{supp}(\alpha)=I} (\alpha)^{\nu_\alpha}.$$

Thus, again using Lemma 3.30, we finally obtain:

$$\gamma_I(\beta')(m) = (m) - \sum_{\mathrm{supp}(\alpha)=I} \nu_\alpha \cdot (\alpha).$$

As above, the fact that $\beta \in \ker(\gamma)$ ensures us that $\gamma_I(\beta)(m) = (m)$ and this implies $\nu_\alpha = 0$ for any (α) of support I. Repeating the same argument for any $I \subset \{1, \ldots, n\}$ of length $k+1$, we get that $\nu_\alpha = 0$ for any α of degree k, which concludes the proof. $\square$

Corollary 3.32. *The normal form is unique in $\tilde{B}_n$; i.e. if*

$$\beta = \prod (\alpha)^{\nu_\alpha} = \prod (\alpha)^{\nu'_\alpha}$$

are two normal forms of β for a given order on the set of twisted comb-claspers, then $\nu_\alpha = \nu'_\alpha$ for any α.

Proof. See [6, Corollay 3.34]. □

Conclusions

As seen in Corollary 3.32, the faithfulness of the representation 3.23 implies that the normal form of a homotopy braid is unique. Thus, the numbers ν_α of parallel copies of each comb-clasper in a normal form are a complete invariant of pure braid up to homotopy. Those numbers are called *clasp-numbers*. In [7] the authors discuss other well known complete homotopy braid invariants such as the *Milnor numbers*: a natural question is to determine (whether it exists) an explicit relationship between these two families of invariants.

Recall that these clasp-numbers depend on the chosen order on $\mathcal{F}$: it could be interesting to study the behaviour of the construction for different orderings on $\mathcal{F}$.

Another faithful representation is given in [11]. Here, the author makes use of it in order to show that the homotopy braid group $\tilde{B}_n$ is torsion-free for $n \leq 6$. However, Humphries representation has greater dimension than the one we studied and the correspondence between them has not been established yet. A further development of this work could be the use of $\gamma : \tilde{B}_n \to GL(\mathcal{V})$ for an in-depth analysis of the torsion problem for more than six strands.

Beyond that, as mentioned above, the study of braids appears to be a valid support to the study of links up to both isotopy and link-homotopy and surely it must not be confined to the world of hairstyling, as most people who read this thesis (such as friends and relatives) may think.

Acknowledgements

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