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Aperiodic Tilings

Master Thesis
in Advanced Geometry

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Contents

Introduction

In this thesis we discuss the problem of plane tessellations, also known as tilings, with a focus on non-periodic tessellations in which there are no translational symmetries. We will therefore study those geometric figures that, if placed next to each other, have the ability to fill the entire plane without overlapping and without leaving empty spaces. We will try to classify them and analyze the possible tilings to which they give rise, using mathematical tools common in the tiling theory.

Since ancient times, the life of man has been accompanied by the art of tessellation, we can find the first examples in the mosaics of the civilization of ancient Mesopotamia which later spread to the Greeks and Romans until arriving to the present day with the works of Maurits Cornelis Escher. This type of art, initially born as a need for constructions and decorations, has slowly spread also in the scientific field. In fact, we can find countless examples of tessellations even in nature. Just think of beehives, plant tissues or crystals and snowflakes. Therefore, having to cover entire surfaces, using simple figures and repeated patterns is a universal need, intrinsic in nature.

For this reason, we began to study tessellations from a mathematical point of view, trying to classify them and understand what the minimum requirements were that the tiles must have. From an initial study regarding the simplest tilings that present regularity and repetition, we arrived at the study of the non-regular or non-periodic ones, where there is no set of tiles that can be repeated to construct the entire tessellation.

From Hao Wang's conjecture in the early 1960s about the non-existence of a set of aperiodic tiles, that is, tiles having the property of tiling the plane only in a non-periodic way, we have come to the discovery of these special sets of tiles thanks to

mathematicians such as Robert Berger, Raphael M. Robinson, Robert Ammann and Roger Penrose who discovered the smallest set of aperiodic tiles known. In fact, he reduced the number of tiles of Berger's set, consisting of 20426 tiles, to only 2 tiles. But recently, just two years ago, a team of researchers discovered the smallest ever aperiodic set of tiles, that consists of a single tile, called the "hat polykite".
Here we will introduce the mathematical tools needed to study the tessellations, then we will move on to the study of polygonal tessellations. We will first analyze the periodic ones, then we will focus on the non-periodic case. We will study the aperiodic Penrose set and finally we will present the proof regarding the aperiodicity of the hat polykite given by the team in the paper "An aperiodic monotile".
This thesis is divided into four chapters.

- **Chapter 1**: In the first chapter we introduce the fondamental tools of plane Euclidean Geometry needed to study the tilings. We start from the Hilbert's axioms, defining all the fundamentals geometric objects like segments, semi-planes, angles, vectors and the concept of measure of segments. Then we focus on the affine transformations of the plane, proving the uniqueness and existence theorem for affinities, and classifying them with respect to the set of their fixed points. This allows us to introduce isometries and similarities which are fundamental tools for the study of tessellations, proving that there are only four types of plane isometries.

- **Chapter 2**: In the second chapter we present the tiling theory, giving the fundamental definitions about tilings, symmetries and periodicity. Then we study the tassellation by regular polygons, analyzing regular, semiregular and Laves tilings. The last type of tilings will then be necessary to introduce the hat polykite. The chapter ends with a section regarding the problem of existence of tilings, where we introduce some techniques necessary for the non-periodic case, proving the Extension theorem.

- **Chapter 3**: In the third chapter we talk about the Penrose aperiodic set P_2, giving a proof of its aperiodicity following the technique used by Robinson and making use of the tools introduced in chapter 2.

- **Chapter 4**: In the fourth chapter we present the first example of an aperiodic monotile, the hat polykite, following all the steps of the proof given by the team in the paper “An aperiodic monotile”. The chapter is divided into three sections. In the first section we describe the hat, showing the properties observed by the authors. Then in the second and third sections we continue with the proof that make use of a novel approch in the tiling theory.

Chapter 1

Basic notions of Euclidean Geometry

In this chapter we recall some aspect of plane Euclidean Geometry that will be needed to understand the following arguments of this thesis. In particular, we will focus our attention on affine transformations, isometries and similarities.
The Euclidean Geometry, developed by Euclid in 300 BC, is based on the assumption of five axioms about points, lines and circumferences. In the late 19th century, David Hilbert proposed a rigorous formulation of this geometry, by introducing a new set of axioms in his work "Grundlagen der Geometrie" (Foundations of Geometry). In formalizing these axioms, Hilbert preserves the core principles of Euclidean Geometry, but also discuss their independence. Six primitive notions, constitute the foundations of his axiomatic system, there are three terms: *point*, *line* and *plane* and three relations: *betweenness*, *lying on* and *congruence*. Hilbert formulate a set of axioms that precisely describe the relationships between those notions. The axioms are designed to describe the fundamental properties of Euclidean Geometry, such as the existence and uniqueness of lines through two points, the ordering of points on a line through betweenness, and the "equality" of lengths and angles through congruence.

1.1 The Hilbert axioms

Hilbert's axioms are subdivided into five categories, each focusing on a specific aspect of geometry. A reduction of these axioms to plane geometry is shown in Appendix A.

The first category is composed of the *axioms of incidence* which describe the relationships between points, lines, and planes. Using these axioms and the lying on relation it is possible to introduce the notion of *parallelism* between lines. Given two lines in the plane, we will say that they are parallel if they have no common points, incident if they have only one point in common, otherwise they coincide. According to the axioms, these are the only three possible cases for the arrangement of two lines in the plane. It is convenient to consider parallelism in a weak sense, that is, to call that two lines are parallel if they are disjoint or coincident.

The second category concerns the *axioms of order*, which formalize the concept of order along a line in terms of the betweenness relation. They also allow to define various fundamental geometric objects such as: ray, segment, semi-plane and angle. We now see how to define these objects.

Given a line r and a point $O \in r$, we can define an equivalence relation on $r - \{O\}$ as follows: for $P, Q \in r - \{O\}$ we say that P is equivalent to Q, and write $P \sim Q \iff O$ is not between P and Q. This equivalence relation has two classes of equivalence that we call *rays* with origin O.

To define a *segment* we need two points P, Q on a line r, then the segment $\overline{PQ}$ is the set of points $\{P, Q\} \cup \{R \in r \mid R \text{ is between } P \text{ and } Q\}$, with end points P, Q. If $P = Q$ then we have a null segment that reduces to a single point. We can also define the notion of *oriented segment* by ordering the end points. We denote with $\overrightarrow{PQ}$ the oriented segment, where P is the starting point and Q is the final point. Note that $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are not the same oriented segment because they have opposite orientation. In the first case we write $P < Q$ while in the second case $Q < P$.

We can extend this concept of order to lines as well, defining the *oriented line*. The betweennes relation gives two possible opposite orientations, that is, there are two total order relation for the points of a line. A point $S \in r$ is between $R, T \in r \iff$

$R < S < T$ or $T < S < R$. These two total orders are induced by the choice made for two distinct points P and Q belonging to r, that is $P < Q$ or $Q < P$.
Let's now define the *semi-plane*. Given a line r, we can define the following equivalence relation between points of the plane excluding r: $P \sim Q \iff \overline{PQ} \cap r = \emptyset$ that is the points P and Q are equivalent if and only if the segments $\overline{PQ}$ does not meet the line r. Pasch's axiom ensures that this relation is an equivalence relation with two equivalence classes. Thus the complement of the line r in the plane is divided into two equivalence classes that we call semi-plane with origin r.
Lastly we can define convex and concave *angle*. Consider two rays s, t with the same origin O. Four semi-planes $\sigma, \sigma', \tau, \tau'$ are associated with these two rays, the first two originate from the line containing s and the other two from the one containing t. Let $s \subset \tau$ and $t \subset \sigma$, then the convex angle $\widehat{st}$ is the intersection between the semi-planes σ and τ, while the concave angle is the union between σ' and τ'. The point O is called vertex of the angle while the rays s, t are called sides of the angle. By ordering the sides we can give an orientation to the angle. Only two orientations are possible $\widehat{st}$ and $\widehat{ts}$. Note that, even though they describe the same set of points, the oriented angles $\widehat{st}$ and $\widehat{ts}$ are not the same angle, because they have opposite orientation.
The third category of Hilbert's axioms consists of a single axiom: the *parallelism axiom* that rephrases Euclid's parallel postulate. It states that, given a line l and a point $A \notin l$, there is exactly one line that passes through A and does not intersect l. So it guarantees the existence and uniqueness of a line parallel to a given line passing through a point not belonging to it.
This axiom allows us to define an equivalence relation called the equipollence relation between oriented segments. We say that two oriented segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are elementary equipollent, if they satisfy: $\overrightarrow{PQ} \parallel \overrightarrow{RS} \wedge \overrightarrow{PR} \parallel \overrightarrow{QS}$. Instead we say that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are equipollent if there is a finite chain of elementarily equipollences between $\overrightarrow{PQ}$ and $\overrightarrow{RS}$, in which case we write $\overrightarrow{PQ} \equiv \overrightarrow{RS}$. Using this relation we can introduce the notion of geometric vector.
We call the *applied vector* $\overrightarrow{PQ}$ at point P, the oriented segment with ordered end points P and Q, $P < Q$. We also define the *free vector* $v = [\overrightarrow{PQ}]$ as the equipollence

class of the applied vector $\overrightarrow{PQ}$. We can verify that for every free vector v and any point P there is a unique point Q such that $v = [\overrightarrow{PQ}]$.
We will now show how it is possible to multiply and divide an oriented segment on a line, using two geometric constructions that are based on the use of the axioms introduced so far. This will then allow us to compare two parallel oriented segments, associating a rational number to their ratio. In the case where two segments are equipollent their ratio is 1.

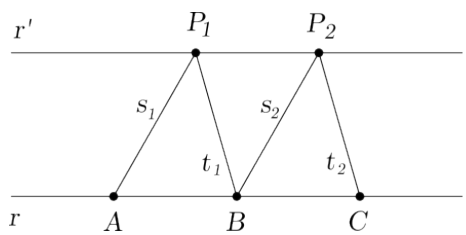


Figure 1.1: The picture shows how to double the oriented segment $\overrightarrow{AB}$.

First let's see how to double an oriented segment. Let $\overrightarrow{AB}$ be an oriented segment on a line r as shown in figure 1.1. Consider a point $P_1 \notin r$ and draw the unique line r' parallel to r and passing through P_1. Starting from A, consider the line s_1 passing through A, P_1 and draw the unique line s_2 parallel to s_1 and passing through B. Call P_2 the intersection point between r' and s_2. By construction $\overrightarrow{AB} \equiv \overrightarrow{P_1P_2}$. Now repeat the same method for B, consider the line t_1 passing through B and P_1 and draw the parallel line t_2 to t_1 passing through P_2. Call C the intersection point between r and t_2. Again by construction $\overrightarrow{P_1P_2} \equiv \overrightarrow{BC}$. Since the equipollence relation is an equivalence relation, then $\overrightarrow{AB} \equiv \overrightarrow{BC}$, moreover they are adjacent segments that share the point B. If we consider $\overrightarrow{AC}$, it is made up with two adjacent equipollent segments, so the ratio $\overrightarrow{AC}/\overrightarrow{AB} = 2$.
By repeating this procedure, we can construct all the integer multiples of a segment. It will be enough to continue to identify m points $P_1, P_2, ..., P_m$ on the line r' using the parallels as just shown. Note that in order to construct negative integer multiples, we first have to invert the oriented segment $\overrightarrow{AB}$. We draw the line t_1 through P_2 and A and its parallel line t_2 passing through P_1 intersecting r on a point C. In this

way we get the oriented segment $\overrightarrow{AC}$ that is in a ratio of -1 with $\overrightarrow{AB}$. Then to get any negative multiple of $\overrightarrow{AB}$, we apply the same method as before to $\overrightarrow{AC}$.
We can now see how to divide a segment into n equal parts with n a positive integer. Let's start by seeing how to divide the segment into 2 parts.

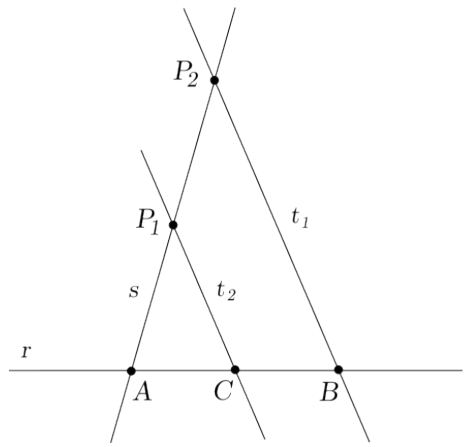


Figure 1.2: The figure shows how to divide the segment $\overrightarrow{AB}$ in two equal parts. By constructing n contiguously copies of $\overrightarrow{AP_1}$ on s is possible to divide $\overrightarrow{AB}$ in n equal parts.

Let $\overrightarrow{AB}$ be an oriented segment on a line r as shown in figure 1.2. Consider a point $P_1 \notin r$ and draw the line s passing through A, P_1. Using the prevoius method, double the oriented segment $\overrightarrow{AP_1}$ on s. We get the unique point $P_2 \in s$ such that $\overrightarrow{AP_1} \equiv \overrightarrow{P_1P_2}$. Now consider the line t_1 passing through B, P_2 and draw the unique line t_2 parallel to t_1 and passing through P_1. Call C the intersection point between r and t_2 that lies on $\overrightarrow{AB}$ (guaranteed by Pasch's axiom since $t_1 \parallel t_2$, so they can't intersect). Using the weak version of Thales' theorem, we have that if $\overrightarrow{AP_1} \equiv \overrightarrow{P_1P_2}$ then also $\overrightarrow{AC} \equiv \overrightarrow{CB}$, because t_1 and t_2 are two parallel lines intersected by the transversals r and s. So we have divided $\overrightarrow{AB}$ into two equal parts. In this case we have that $\overrightarrow{AC}/\overrightarrow{AB} = \frac{1}{2}$.
If we want to divide the segment into n equal parts, then we have to construct n points $P_1, P_2, ..., P_n$ on the line s such that $\overrightarrow{AP_1} \equiv \overrightarrow{P_1P_2} \equiv ... \equiv \overrightarrow{P_{n-1}P_n}$ and then starting from P_n we have to draw all the parallel lines to the line connecting P_n to

B as just shown.
Starting from a given segment $\overrightarrow{AB}$, is then possible to construcut a new segment $\overrightarrow{AC}$ in a ratio $\frac{m}{n}$ with $\overrightarrow{AB}$ by using these two geometric methods. We just need to first multiply $\overrightarrow{AB}$ by a factor m and then divide the resulting segment $m\overrightarrow{AB}$ by a factor n. In other words, we are able to construct any rational multiple of a given segment, and consequently we are also able to compare two weakly parallel segments, if they are commensurable, that is they share a common submultiple.
Imagine we have two oriented segments $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ that lie on two parallel lines and to want to calculate the ratio between them, that is $\overrightarrow{PQ}/\overrightarrow{AB}$. We first need to construct the equipollent segment $\overrightarrow{A'B'}$ to $\overrightarrow{AB}$ on the line that contains $\overrightarrow{PQ}$ in such a way that the point A' coincides with P. This can be done by drawing the line passing through A and P ($A' = P$) and its parallel line passing through B (B' is the intersection point on the line containing $\overrightarrow{PQ}$). Then we can extend $\overrightarrow{A'B'}$ to macth $\overrightarrow{PQ}$ and count how many times it fits into it. If $\overrightarrow{A'B'}$ fits exactly a finite number of times into $\overrightarrow{PQ}$, then we have found the ratio, otherwise we have to consider the submultiples of $\overrightarrow{A'B'}$ and check if one of them fits exactly a finite number of times into $\overrightarrow{PQ}$. Only in the case that the ratio is rational, we will be able to find the right submultiple that must be a common one to the two segments and the procedure ends. Otherwise this procedure will never end for any submultiples. If instead $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ lie on the same line r, we proceed in the same way, except for the first step which must be done in two steps using a chain of elementarily equipollences.
Note that this method allows us to calculate the ratio of two weakly parallel segments only if it is a rational number. In order to compare any pair of weakly parallel segments, we must introduce the next category of axioms, the so-called *continuity axioms*. They allow us to construct a multiple of a given segment with any real factor, so we can associate a real value to the ratio between two weakly parallel segments even when these are not commensurable.
Let's first see how to construct a real multiple of a given segment. Let $\overrightarrow{AB}$ be an oriented segment on a line r and k a real number. Our goal is to construct $k\overrightarrow{AB}$. Consider a sequence of rational numbers that converges to k. For each rational number in the sequence let construct the multiple corresponding to that rational

number in such a way that the first end point coincides with A (just use the methods shown before to multiply and divide a segment). The second end point depends on the multiple we consider. The continuity axioms tell us that these second end points converge to a limit point. This limit point is the second end point of $k\overrightarrow{AB}$.

So now we are able to define the ratio between any two weakly parallel segments $\overrightarrow{PQ}$ and $\overrightarrow{AB}$, that is $\overrightarrow{PQ}/\overrightarrow{AB}$. As before we construct the equipollent segment $\overrightarrow{A'B'}$ to $\overrightarrow{AB}$ on the line that contains $\overrightarrow{PQ}$ in such a way that the point A' coincides with P. Then their ratio is the unique real number k for which the second end point of $k\overrightarrow{A'B'}$ coincide with Q.

From the continuity axioms follows that, given a segment $\overrightarrow{AB}$ on a line r, for any point $P \in r$ there is a unique real number k such that the second end point of $k\overrightarrow{AB}$ is P and k is the ratio between the two segments.

The fifth and last category of Hilbert's axioms concerns the *congruence axioms*. They establish rules for comparing and measuring segments and angles. From these axioms one can derive the three congruence criteria of triangles and the fact that for any point P on a line r, there is a unique line r' passing through P perpendicular to r, that is, it forms 4 congruent angles.

The first congruence axiom ensures that, given a segment $\overline{AB}$ on a line r, and a point A' on the same or on another line r', it is always possible to find a point B' on a given side of the line r' through A' such that $\overline{AB}$ is congruent to $\overline{A'B'}$. So on r' there are two rays with origin A', each of which contains a points B' such that $\overline{AB} \cong \overline{A'B'}$. From this it follows that we can calculate the ratio between segments even when these are not weakly parallel. Indeed if $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are two oriented segment that belong to any two lines, it's always possible to construct a congruent segment $\overrightarrow{A'B'}$ to $\overrightarrow{AB}$ on the same ray with origin P that contains $\overrightarrow{PQ}$ where $A' = P$.

We can now introduce the concept of measure of segments. Let $\overline{OU}$ be a segment on a line r, it will be our *unit of measurement* and define the measure of $\overline{OU}$ equal to 1. In symbols: $|\overline{OU}| = 1$. From the first axiom we know that, given $\overline{PQ}$ on r', there is a unique point $R \in r'$ such that $\overline{OU} \cong \overline{PR}$ and R lies on the ray originated by P that contains Q. With these conditions we can define $|\overline{PQ}| = \overrightarrow{PQ}/\overrightarrow{PR}$. Note that this measure is always a positive number and is equal to zero if and only if $P = Q$.

Moreover, $\overline{PQ} \cong \overline{RS}$ if and only if $|\overline{PQ}| = |\overline{RS}|$.

1.2 Affine Transformations

As said at the beginning, we are interested in the transformations of the plane, in particular in those transformations that have some geometric properties. We will indicate with $\mathbb{E}^2$ the Euclidean plane, while we will use the notation $\mathbb{A}^2$ for the affine plane, where congruence relation is disgarded.
We can start by giving the general definition of an affine transformation, usually called *affinity*.

Definition 1 (Affine Transformation) *An affine transformation is a bijective application $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ that maps any line to a line.*

The affinities of the plane form a group respect to the composition between applications, that is usually indicated with Aff. Indeed, it is always possible to compose two affinities and their composition is an affinity because any line is sent to a line by both applications. The neutral element of the group is the identity application. Regarding the inverse, we need to verify that the inverse of an affinity is an affinity. Let φ be an affinity and A', B', C' three aligned points. We have to show that their counter-images $A = \varphi^{-1}(A')$, $B = \varphi^{-1}(B')$ and $C = \varphi^{-1}(C')$ are aligned, so φ^{-1} sends a line to a line, thus is an affinity. Consider the line r passing through A and B, it is mapped by φ to the line r' passing through A' and B'. By hypothesis C' is aligned with A' and B', so it belongs to r'. Since φ is a bijection, it maps intersections into intersections of the images. So if C does not belong to r, that is $\{C\} \cap r = \emptyset$, then $\{C'\} \cap r' = \emptyset$ which is absurd. So A, B, C are aligned, thus φ^{-1} is an affinity.
An affinity has a lot of geometric properties all deriving from its definition. One of this is that an affinity transforms three non-aligned points into three non-aligned points, in other words it maps a triangle into a triangle, more precisely the following proposition holds.

Proposition 1 *A map $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ is an affinity $\Longleftrightarrow$ φ is a bijection and maps any triple of non-aligned points, into a triple of non-aligned points.*

Proof.
By definition an affinity sends any line to a line, meaning that it sends any three aligned points into three aligned points. So by contradiction, φ send three non-aligned points into three non-aligned points. In fact if φ sent three non-aligned points into three aligned points, then φ^{-1} would send three aligned points into three non-aligned points, which is absurd. $\square$

A very important property of an affinity is that it preserves the parallelism between lines, because the notion of parallelism is based on the intersections. In fact, since an affinity is a bijection, two lines have empty intersection if and only if their images have empty intersection.
A direct conseguence of this property is that an affinity preserves the equipollence relation, if $\overrightarrow{PQ} \equiv \overrightarrow{RS}$ then also $\varphi(\overrightarrow{PQ}) \equiv \varphi(\overrightarrow{RS})$. This comes from the fact that affinities preserve parallelism and that the equipollence relation is defined in terms of parallelism.
It follows that an affinity preserves the ratio between weakly parallel segments. Recall the two geometric constructions we used in the previous section to multiply and divide a segment, shown in figures 1.1 and 1.2. On both occasions, we have only used lines, parallelism and the equipollence relation, that is, features conserved by an affinity. This means that if we apply any affinity to these constructions we get the multiplication and division of the transformed segments.
This implies that even in the case of two incommensurable weakly parallel segments, an affinity preserves their ratio, because in the previous proof (the one in the paragraph concerning the axioms of continuity) we only used these two constructions which are preserved.
We can now prove the uniqueness and existence theorem of an affinity.

Theorem 1 *For any two triples $A, B, C \in \mathbb{A}^2$ and $A', B', C' \in \mathbb{A}^2$ of non-aligned* *points, there is a unique affinity $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ such that $A' = \varphi(A), B' = \varphi(B)$ and $C' = \varphi(C)$.*

Proof.
(Uniqueness)
Let $A, B, C \in \mathbb{A}^2$ and $A', B', C' \in \mathbb{A}^2$ be non-aligned points. Denote with r and r' the lines between A, B and A', B', and with s and s' the lines between A, C and A', C'. Now take a point $D \in r$ and a point $E \in s$. In order to construct the images of D and E, let's suppose that φ exists. We can use the fact that φ has the property of preserving the ratio between two weakly parallel segments. Consider the segments $\overrightarrow{AB}$ and $\overrightarrow{AD}$ on r. They are weakly parallel because they both lie on the same line, so there is a unique point $D' \in r'$ that satisfies $\overrightarrow{AD}/\overrightarrow{AB} = \overrightarrow{A'D'}/\overrightarrow{A'B'}$. We can also apply this process to the line s to find the unique point $E' \in s'$ that satisfies $\overrightarrow{AE}/\overrightarrow{AC} = \overrightarrow{A'E'}/\overrightarrow{A'C'}$.

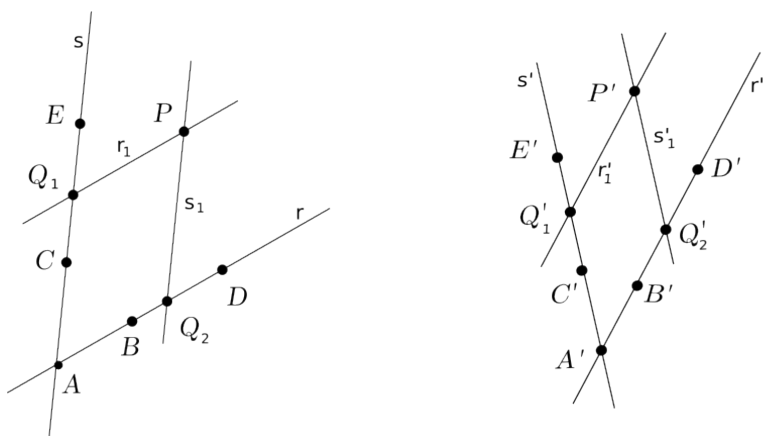


Figure 1.3: The figure shows an example of a geometrical construction necessary for the uniqueness of an affinity.

Now we want to contruct the image of a generic point P of the plane. We can do this by using the fact that an affinity preserves parallelism between lines. Consider any point P and draw from this point two parallel lines r_1 and s_1 to r and s so that $P \in r_1 \cap s_1$ as shown in figure 1.3. Let Q_1 be the point of intersection between r_1 and s, and Q_2 the intersection point between s_1 and r. Use the same method as before

to calculate the images of Q_1 and Q_2 on s' and r'. From Q_1' and Q_2' draw two lines ${r_1}'$, ${s_1}'$ parallel to r' and s'. They must be the images of r_1 and s_1. In fact, since an affinity preserves parallelism, r_1 has to be mapped to a line parallel to r' that pass for Q_1' and the same is true for s_1 with a line parallel to s' that pass for Q_2'. So their intersection must be the image of P under the action of φ. This guarantees that φ is unique, because the image of a generic point P was obtained in a unique way starting from the sole knowledge of A, B, C and their images.
(Existence)
Lastly we have to show that this unique possible φ defined with the construction above of a generic point P, is actually an affine transformation, that is, it must send any line into a line. Without going into the details of the proof, we limit ourselves to saying that it is possible to show this using Thales' theorem. $\square$

We now want to define some important affine transformations.
Given a free vector v, we define the *translation* by the vector v, $\tau_v : \mathbb{A}^2 \to \mathbb{A}^2$ as the application that maps each point P to the point P' such that $v = [\overrightarrow{PP'}]$. If v is the null vector, then τ_v is the identity.
A translation sends an oriented segment to an oriented segment equipollent to it. Let's consider the generic case in which an oriented segment $\overrightarrow{PQ}$ and the free vector $v \neq 0$ are not parallel. The special case in which $\overrightarrow{PQ} \parallel v$ can be treated separately and it is simpler. By applying the translation τ_v to P and Q, we get the points P' and Q' such that $v = [\overrightarrow{PP'}] = [\overrightarrow{QQ'}]$. Clearly $\overrightarrow{PP'} \equiv \overrightarrow{QQ'}$ and the points P, Q, P', Q' forms a parallelogram because P, Q, P' are not aligned. So $\overrightarrow{PP'} \parallel \overrightarrow{QQ'}$ and $\overrightarrow{PQ} \parallel \overrightarrow{P'Q'}$, from this it follows that $\overrightarrow{PQ} \equiv \overrightarrow{P'Q'}$.
So the translations preserve the equipollence relation.
Another example of affine transformation is given by the so-called *dilatations*. Given a point O (called the center) and a real number k (called the dilatation factor), we define the dilatation of center O and factor k, $d_{O,k} : \mathbb{A}^2 \to \mathbb{A}^2$ as the application that maps each point P to a point P' such that $\overrightarrow{OP'} = k\overrightarrow{OP}$.
We can show that a dilatation is indeed an affine transformation. Let P, Q, R be

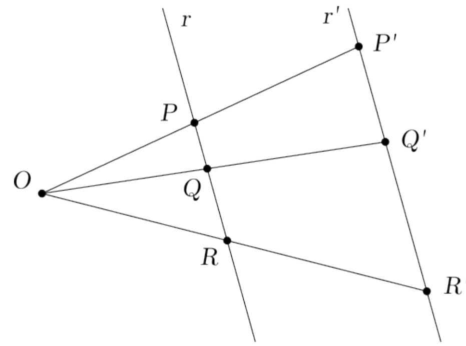


Figure 1.4: The figure shows the action of a dilatation on three aligned points P, Q, R. Their images P', Q', R' are aligned thanks to the inverse Thales' theorem.

three aligned points, we want to show that their images P', Q', R' are aligned, so that $d_{O,k}$ sends any line into a line. Let r be the line that contains P, Q, R as shown in figure 1.4. Clearly if $O \in r$ then by definition P', Q', R' belong to r as well. So let's suppuse that $O \notin r$ and apply $d_{O,k}$ to these points. Then holds that: $\overrightarrow{OP'}/\overrightarrow{OP} = \overrightarrow{OQ'}/\overrightarrow{OQ} = \overrightarrow{OR'}/\overrightarrow{OR} = k$, so for the inverse Thales' theorem P', Q', R' must belong to the line r' parallel to r, thus $d_{O,k}$ is an affinity.
Sometimes it may happen that an affine transformation sends a point P to itself, in this case P is called a *fixed point* of the affinity. We denote the set of all fixed point of an affinity φ by:

$$\mathrm{Fix}(\varphi) = \{P \in \mathbb{A}^2 \mid \varphi(P) = P\}$$

Theorem 2 *Let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be an affine transformation, the set $\mathrm{Fix}(\varphi)$ can be one and only one of the following: the empty set, a point, a line or the plane.*

Proof.
Let's first see that those listed are the only possibilities for $\mathrm{Fix}(\varphi)$. Then we will show that they are actually realizable, that is, there is an example of an affine transformation for each of them. The first two cases are trivial. $\mathrm{Fix}(\varphi)$ is a set, so it can be empty or it can contain one point. If instead there is also another point in it, then it contains the entire line passing through these two points. Indeed let $P, Q \in \mathrm{Fix}(\varphi)$ and r be the line passing through them. Consider another point

$R \in r$ and apply φ to all three points. Since φ is an affinity it must hold that $\overrightarrow{P'R'}/\overrightarrow{P'Q'} = \overrightarrow{PR}/\overrightarrow{PQ}$ because they are weakly parallel segments. But P and Q are fixed point, so $P' = P$ and $Q' = Q$, then holds that $\overrightarrow{PR'}/\overrightarrow{PQ} = \overrightarrow{PR}/\overrightarrow{PQ}$, which is true if and only if $R' = R$. Thus $\mathrm{Fix}(\varphi)$ contains the entire line r. If in $\mathrm{Fix}(\varphi)$ there is another point outside of r, then it contains the entire plane. Let $A, B, C \in \mathrm{Fix}(\varphi)$ be three non-aligned points, then from the uniqueness and existance theorem for the affinity, there is only one affinity such that $A = \varphi(A)$, $B = \varphi(B)$ and $C = \varphi(C)$. The identity satisfies this relation for any point of the plane, then $\mathrm{Fix}(\varphi)$ contains the entire plane.

An example of an affine transformation that has no fixed points is the translation τ_v. Indeed $\tau_v(P) = P$ if and only if $v = [\overrightarrow{PP}]$, that is v is the null vector, but in this case the translation coincides with the identity. So for any non zero vector v, $\mathrm{Fix}(\tau_v) = \emptyset$.

Regarding the case where $\mathrm{Fix}(\varphi)$ has only one point, we can consider the dilatations $d_{O,k}$ of center O and factor $k \neq 1$. The only fixed point is the center O. Indeed by definition $d_{O,k}(O) = O'$, where $\overrightarrow{OO'} = k\overrightarrow{OO}$. But $\overrightarrow{OO}$ is a null segment that coincides with the point O, so $\overrightarrow{OO'}$ has to be a null segment as well, that is $O' = O$. For any other point P, the relation $\overrightarrow{OP} = k\overrightarrow{OP}$ holds if and only if $k = 1$. So any dilatation $d_{O,k}$ with $k \neq 1$ fixes only its center O.

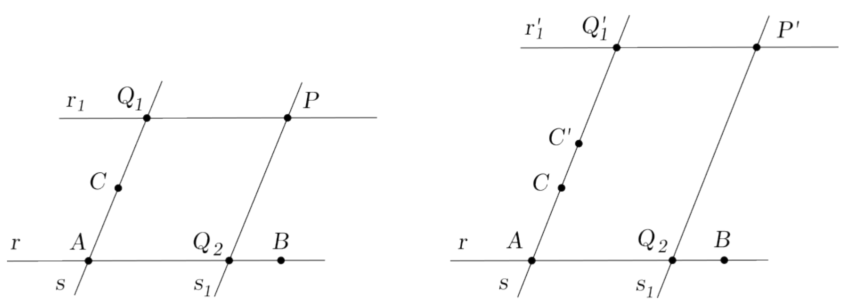


Figure 1.5: The figure shows an example of an affine transformation that fixes a line.

In order to show an example of an affine transformation that fixes a line, we can define φ by its action on three non-aligned points. Let A, B, C be three non-aligned points, then define $\varphi(A) = A$, $\varphi(B) = B$ and $\varphi(C) = C'$, where $C' \neq C$ and

belongs to the line passing through A and C as shown in figure 1.5. Let r be the line containing A, B and s the line containing A, C. As we have already shown, φ must fix the entire line r because A, B are fixed points. Consider a point $P \notin r$ and apply φ to it. We can determine the image of P by doing the same construction used in the proof of uniqueness of an affinity. From P we draw the parallel lines r_1, s_1 to r, s and consider the intersection points Q_1 and Q_2 respectively. We then construct the images of Q_1 and Q_2 using the ratio of parallel segments as we have done previously. From these points we draw the parallel lines r_1', s_1', so the image of P is uniquely determinated by their intersection. In this case we have that, $Q_2' = Q_2$ because Q_2 belongs to the fixed line r, while $Q_1' \neq Q_1$ and belong to s since $A, C, C' \in s$. So the line s_1 is mapped to itself (although it is not fixed), while r_1 is mapped to a distinct line r_1' parallel to it, thus P' must be distinct to P. This implies that φ only fixes the line r.
Regarding the last case, we have already given an example of an affine transformation that fixes the entire plane, it is the identity. □

1.3 Isometries and similarities

Now we introduce a special type of affinity, that have the property of preserving the congruence relation, that is it maps any segment to a segment congruent to it. These transformations are called *isometries* and are fundamental in understanding the symmetries and rigid motions of objects in a given space.

Definition 2 (Isometry) *An affinity $f : \mathbb{E}^2 \to \mathbb{E}^2$ is an isometry if for any $A, B \in \mathbb{E}^2$*

$$\overline{AB} \cong \overline{f(A)f(B)}$$

Alternatively one could also define isometries without assuming that they are affinities, because this follows from the fact that an isometry sends a segment into a segment congruent to itself. In fact three points A, B, C with B between A and C are aligned if and only if $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$. By applying an isometry one has that $|\overline{AB}| = |\overline{A'B'}|$, $|\overline{BC}| = |\overline{B'C'}|$ and $|\overline{AC}| = |\overline{A'C'}|$, therefore it holds $|\overline{A'C'}| = |\overline{A'B'}| + |\overline{B'C'}|$. So also the images are aligned.

The isometries of the plane form a subgroup of the affinities respect to the composition denoted with Iso. This comes from the fact that the composition of isometries is an isometry. Indeed let f and g be two isometries and $\overline{AB}$ a segment. By definition $\overline{AB} \cong \overline{f(A)f(B)}$, if we compose f with g, we get that $\overline{f(A)f(B)} \cong \overline{g(f(A))g(f(B))}$. Since the congruence relation is an equivalence relation then $\overline{AB} \cong \overline{g(f(A))g(f(B))}$, so $g \circ f$ is an isometry.

Euclidean isometries are divided into *positive* isometries that preserve the orientation of the plane, that is, the orientation of the angles, and *negative* isometries that invert the orientation. We can observe that if we combine a positive isometry with a negative isometry, we get a negative isometry, because one preserves the orientation of the angles while the other invert the orientation. But if we apply another negative isometry to this composition, we get a positive one. So positive and negative isometries combine like positive and negative numbers combine in multiplication.

Since an isometry is an affinity, it must satisfy all the properties discussed above. In particular it must send a triple of non-aligned points A, B, C (that we can call a *triangle*), into another triple of non-aligned points A', B', C' where $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$ and $\overline{AC} \cong \overline{A'C'}$. Therefore the triangle $\triangle ABC$ must be congruent to $\triangle A'B'C'$ for the third criterion of congruence of triangles. From this it follows that isometries also preserve the angles.

Let's now define all four types of isometries of the plane which we will then show to be the only possible ones. Let's start by defining the reflections.

A *reflection* is totally determined by a line m, usually called *mirror*. We define the reflection with respect to m, $\sigma_m : \mathbb{E}^2 \to \mathbb{E}^2$ as the application that maps any point P to a point P' where m is the axis of the segment $\overline{PP'}$. Given a line m and a point P, one can find P' by drawing the perpendicular line r from P to m, that intersects m on a point Q. Then P' is uniquely defined by the relation $\overrightarrow{QP'} = -\overrightarrow{QP}$. The reflections invert the orientation of the angles.

If $P \in m$ then P' coincide with P since the segment $\overline{PQ}$ reduces to P itself. So in this case P is a fixed point for the reflection. This implies that every point of m are fixed points for a reflection.

$$\mathrm{Fix}(\sigma_m) = m$$

We can show that a reflection is an isometry.

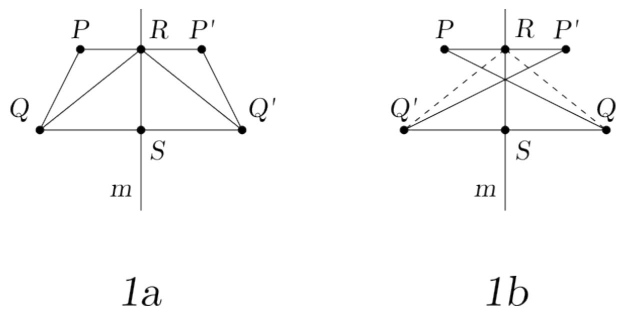


Figure 1.6: The figure shows the two generic situation about the reciprocal position between the points P, Q and the line m. In $1a$ the points belong to the same semi-plane identified by the line m, while in $1b$ they belong to distinct semi-planes.

Let m be the mirror of a reflection and P, Q two points of the plane. We have to show that the segment $\overline{PQ}$ is congruent to its image $\overline{P'Q'}$. If both P, Q belong to m, then we don't have to prove anything because they are fixed points. In the generic case we have two situations shown in figure 1.6, where P and Q don't belong to a line perpendicular to m. The special cases where P and Q belong to a line perpendicular to m can be treated separately and are simpler.
In case $1a$, consider the triangles $\triangle RQS$ and $\triangle RQ'S$. The segment $\overline{RS}$ is congruent to itself, $\overline{SQ} \cong \overline{SQ'}$ by definition and $\angle RSQ \cong \angle RSQ'$ because Q, Q' belong to a line perpendicular with m. So for the first criterion of congruence of triangles $\triangle RQS \cong \triangle RQ'S$, in particular $\overline{RQ} \cong \overline{RQ'}$ and $\angle SRQ \cong \angle SRQ'$. Now consider the triangles $\triangle RPQ$ and $\triangle RP'Q'$. We have that $\overline{PR} \cong \overline{P'R}$ by definition, $\overline{RQ} \cong \overline{RQ'}$ and $\angle PRQ \cong \angle P'RQ'$ because they are the difference of the congruent angles $\angle PRS \cong \angle P'RS$ (by definition) and $\angle SRQ \cong \angle SRQ'$. So for the first criterion of congruence of triangles $\triangle RPQ \cong \triangle RP'Q'$, in particular $\overline{PQ} \cong \overline{P'Q'}$.

The case $1b$ follows the same proof as case $1a$, except that $\angle PRQ \cong \angle P'RQ'$ because they are the sum of the congruent angles $\angle PRS \cong \angle P'RS$ (by definition) and $\angle SRQ \cong \angle SRQ'$.

Now we introduce the rotations.

A *rotation* is completely determined by a point C called *center* and an oriented angle α. We define the rotation around C of an angle α, $\rho_{C,\alpha} : \mathbb{E}^2 \to \mathbb{E}^2$ as the application that maps any point P to a point P' such that $\overline{CP} \cong \overline{CP'}$ and $\angle PCP' = \alpha$.

We need to make some observation regarding the angle. First of all if $\alpha = 0$, then P is mapped to itself, so the rotation reduces to the identity, but we can achieve the same result if $\alpha = 2\pi$ or any integer multiple of 2π. So when we talk about a rotation, we consider only angles $\alpha \neq 0 \mod 2\pi$, otherwise we get the identity. The rotations preserve the orientation of the angles.

If $P = C$, then P is mapped to itself because the segment $\overline{CP}$ reduces to C itself. While any other point is mapped to a distinc point P' that belong to the circunference of center C and radious congruent to $\overline{CP}$. So the center C is the only fixed point for the rotation.

We can show that a rotation is an isometry.

Let C be the center of a rotation of an oriented angle α and P, Q two points of the plane. We have to show that the segment $\overline{PQ}$ is congruent to its image $\overline{P'Q'}$. If P, Q are aligned with C ($|\overline{CQ}| = |\overline{CP}| + |\overline{PQ}|$), then their images P', Q' are aligned too with C and $\overline{CP} \cong \overline{CP'}$, $\overline{CQ} \cong \overline{CQ'}$, so $\overline{PQ} \cong \overline{P'Q'}$.

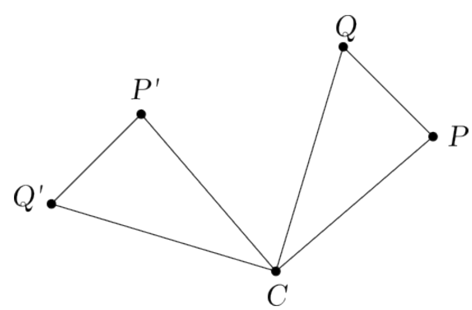


Figure 1.7: The figure shows that a rotation is an isometry, because the triangles $\triangle CPQ$ and $\triangle CP'Q'$ are congruent.

So let's suppose that P, Q are not aligned with C and apply the rotation to them as shown in figure 1.7. We have that the segments $\overline{CP} \cong \overline{CP'}$, $\overline{CQ} \cong \overline{CQ'}$ and the angles $\angle PCP' \cong \angle QCQ'$ by definition. Now consider the triangles $\triangle CPQ$ and $\triangle CP'Q'$. The angles $\angle PCQ$ and $\angle P'CQ'$ are congruent because they are the difference of the congruent angles $\angle PCP' \cong \angle QCQ'$ and the angle $\angle QCP'$ that is congruent to itself. So for the first criterion of congruence of triangles $\triangle CPQ \cong \triangle CP'Q'$, in particular $\overline{PQ} \cong \overline{P'Q'}$.
We have already defined the translations in the section about affinities. The translations preserve the orientation of the angles. We can show that a translation is also an isometry.

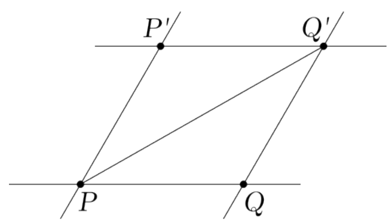


Figure 1.8: The figure shows that a translation is an isometry, because the triangles $\triangle PQQ'$ and $\triangle PP'Q'$ are congruent.

Let τ_v be the translation of a vector v and P, Q two points of the plane. We have to show that the segment $\overline{PQ}$ is congruent to its image $\overline{P'Q'}$.
Regarding the general case shown in figure 1.8, we already know that a translation maps $\overrightarrow{PQ}$ to an equipollent segment $\overrightarrow{P'Q'}$. Consider the triangles $\triangle PQQ'$ and $\triangle PP'Q'$. The segment $\overline{PQ'}$ is congruent to itself, $\angle QPQ' \cong \angle P'Q'P$ because they are alternate internal angles identified by the two parallel lines containing the points P, Q and P', Q' respectively and by the secant line passing through P, Q'. Moreover $\angle Q'PP' \cong \angle PQ'Q$ for the same reason. So for the second criterion of congruence of triangles $\triangle PQQ' \cong \triangle PP'Q'$, in particular $\overline{PQ} \cong \overline{P'Q'}$.
In the special case where v is parallel to the line that contains both P and Q, the congruence between the segments $\overline{PQ}$ and $\overline{P'Q'}$ follows by transitivity using a chain of elementarily equipollences.
This implies also that two equipollent segments are congruent, so the equipollence is

contained in the congruence.

The last type of isometry that we have to define is the so-called *glide reflection*, where a reflection is composed with a translation by a vector parallel to the mirror of the reflection. We define the glide reflection $\eta_{m,v} : \mathbb{E}^2 \to \mathbb{E}^2$ as the application that maps any point P to a point $P' = \tau_v(\sigma_m(P))$.

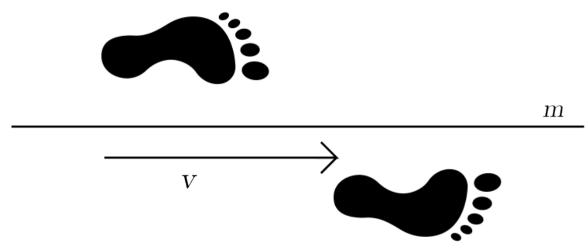


Figure 1.9: The footprints are related by a glide reflection.

A typical example of this transformation can be found in nature quite easily, just think of walking on the beach, the footprints are related by this geometrical transformation. In fact the left and right feet are mirror images of each other and when you walk on the sand they will be separated by a distance as you can see in figure 1.9. The glide reflections invert the orientation of the angles.

Now let's have a look on the set of fixed points for a glide reflection. We know that this transformation is the composition of a reflection and a translation, the first fixes the mirror while the second has no fixed points. Let σ_m be a reflection through a mirror m and τ_v be a translation by a vector $v \neq 0$ parallel to m. If we take a point $P \in m$ and apply σ_m we get P itself, then we apply τ_v and it will be moved to a different point P' along m. So all the points of m will be mapped to different points thanks to the translation. Instead if $P \notin m$ by applying the reflection we already get a point $P' \neq P$ on the other semiplane identified by m and then, when we apply the translation, the point P' remains in this semiplane because v is parallel to m. So like translations, a glide reflection has no fixed points.

Theorem 3 *Every isometry $f : \mathbb{E}^2 \to \mathbb{E}^2$ of the Euclidean plane $\mathbb{E}^2$ satisfies one and only one of the following properties:*

1. $\mathrm{Fix}(f)$ *coincides with the entire plane and therefore* f *is the identity;*

2. $\mathrm{Fix}(f)$ *is a line* r *and* f *is the reflection* σ_r *with respect to this line;*

3. $\mathrm{Fix}(f)$ *is a point* P *and* f *is the rotation* $\rho_{P,\alpha}$ *around* P *of an angle* $\alpha \neq 0 \bmod 2\pi$*;*

4. $\mathrm{Fix}(f)$ *is empty and in this case* f *is a translation (non-zero) if it preserves the orientation or a glide reflection if it inverts the orientation.*

Proof.
We have already observed that the only possible cases for $\mathrm{Fix}(f)$ are those indicated in the statement of the theorem. The first case has already been demonstrated in theorem 2 in the case of affinities, therefore it is also valid for isometries. The second case was discussed in the paragraph about reflections, where we said that a reflection preserves the mirror.
As for case 3, if P is the only point fixed by f, then any circle $\mathcal{C}$ with center P must be transformed into itself while preserving the orientation, otherwise if the orientation of $\mathcal{C}$ were inverted there would exist another fixed point $Q \in \mathcal{C}$ and therefore f would fix the entire line passing through P and Q in contrast with the hypothesis. At this point it is easy to conclude that f is a rotation around P.
Finally, let us consider case 4. Let f be an isometry of the Euclidean plane without fixed points. First of all, we need to prove that there exists a line r that is invariant with respect to f, that is, such that $f(r) = r$. Consider any point $P \in \mathbb{E}^2$ and let $P' = f(P')$ and $P'' = f(P')$, in practice we have applied f recursively to P. If the three points P, P' and P'' are aligned then we can set r as the line passing though P, P'. For simplicity we will write $r = PP'$ and we have $f(r) = P'P'' = r$.
Otherwise one must have one of the two configurations in figure 1.10, where $P''' = f(P'')$. Note that the segments $\overline{PP'}$, $\overline{P'P''}$ and $\overline{P''P'''}$ must be all congruent to each other, by construction. The configuration on the left is impossible because it would imply the existence of a fixed point C. In the case of the configuration on the right, let Q, Q' and Q'' be respectively equal to the midpoints of the segments $\overline{PP'}, \overline{P'P''}$ and $\overline{P''P'''}$, we have that $Q' = f(Q)$ and $Q'' = f(Q')$ with Q, Q' and Q''

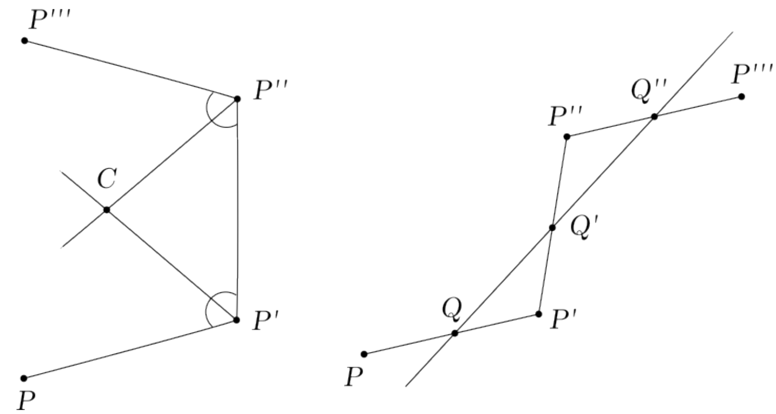


Figure 1.10

aligned, we can then set $r = QQ'$ and as above we have $f(r) = Q'Q'' = r$. Since r is an invariant with respect to f, then f must transform r into itself by a non-trivial translation, otherwise f would fix some point of r. It is now easy to conclude that f is a translation or a glide reflection as required. $\Box$

We can conclude this chapter this introducing another type of affinities that preserves the angles, the so-called *similarities.*

Definition 3 (Similarity) *An affinity $\phi : \mathbb{E}^2 \to \mathbb{E}^2$ is a similarity if for any $O, P, Q \in \mathbb{E}^2$:*

$$\phi(\widehat{POQ}) \cong \widehat{POQ}$$

Alternatively the similarity can be defined as the affinity such that the ratio between any segment and its image is conserved. In symbol: $\phi(\overrightarrow{PQ}) = k\overrightarrow{PQ}$ for any $P, Q \in \mathbb{E}^2$, where k is a positive real number. In fact this definition is equivalent with that given above because a similarity sends any triangle into another triangle with proportional sides. Therefore, by the third criterion of similarity of triangles, we have that the two triangles are similar, that is, they have the same angles.
The real number k is called *similarity ratio.* If $k > 1$, the similarity is a *expansion* whose effect is to move the points of the plane away from each other, vice versa if

$0 < k < 1$, the similarity is a *contraction* and the points will move closer together. The special case where $k = 1$ concerns isometries and has already been discussed.
The similarities also form a group denoted with Sim, which is a subgroup of the affinities, $\mathrm{Sim} \subset \mathrm{Aff}$.
Lastly we can observe that the dilatations contribute to generating similarities, in fact if we compose a similarity by a factor k with a dilatation of a factor $\frac{1}{k}$ we get an isometry. So any similarity of a factor k is the composition of an isometry with a dilatation of a factor k that is the inverse of the one with factor $\frac{1}{k}$.

Chapter 2

Tiling

In this chapter we will talk about the tiling theory, the study of shapes that cover the plane with no gaps or overlaps, examining regular and semi-regular tessellations, the role of symmetry groups, and the problem of aperiodic tilings. From ancient mosaics to modern mathematical research, tiling has fascinated artists, architects, and mathematicians all over the world. There are numerous historical and contemporary examples, ranging from the mosaics found in pre-Columbian civilizations of the Americas, Mesopotamia, ancient Greece, Rome, and Byzantium, to the modern era with the stunning works of Maurits Cornelis Escher. The art of tiling is not just something invented by humans, the nature itself seems to be particulary interessed in findind structures resulting from the combination of elementary objects repeated according to regular patterns and simple rules. One of the most famous example is the hexagonal structure of bee honeycombs. Other examples are the cellular tissues that make up the skin and vegetable leaves, and many more can be find in crystals and snowflakes.

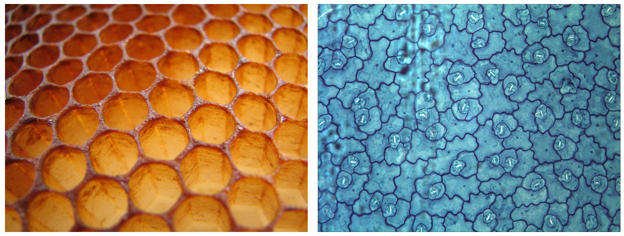


Figure 2.1: The figure shows two examples of real tilings found in nature. On the left we can see a bee honeycombs made up of all regular hexagons, while on the right the surface of a bean leaf under a microscope, in this case the tessellation is made up of all different geometric shapes.

2.1 Basic Definitions

The fundamental characteristic of a plane tiling is that it consists of shapes that completely cover the plane without overlapping. Of course there are uncountably many possible tessellations that range from the most regular to the highly irregular ones, some can be created by arranging copies of just one single shape, others may require two, three or even infinite shapes all different from each other.
Let's begin to give some restrictions to the shapes that we want to consider as *tiles*. First of all we require that our tiles are connected polygons, having a connected simple curve as boundary, and hence having no holes. We don't want those composed of multiple disjoint parts or connected solely at points. In order to satisfy these constraints we require that every tile has to be *topologically equivalent to a closed unit disc*. In this way each tile is a bounded subset of the plane that contains its boundary and encloses a finite area. Moreover, we require our tilings to be *locally finite*, that is every point of the plane has some open neighbourhood that meets only finitely many tiles. This restriction guarantees that every tiling contains precisely a countable infinity of tiles, which we can refer to as $\{ T_1, T_2, ... \}$.
Since our tiles are *closed* we need to make some clarification regarding the intersection of two tiles. For example, just as it's impossible to cover the real line with disjoint connected closed intervals, (the end points of the intervals must necessarily overlap), similarly, disjoint closed topological discs cannot cover the plane without overlapping

along their boundaries. Therefore, we allow tiles to have a non-empty intersection if that intersection occurs within the tiles boundaries. Only when the interiors of tiles intersect we have an overlap.
Another aspect on which we have to give some restrictions concerns the size of tiles. Let T be a tile, since T is bounded, there exists a real number $U_T > 0$ such that T is completely contained in a closed disc of radius U_T. In order to avoid undesirable behaviour at infinity we require all tiles to be uniformly bounded, that is there exist a real number $U > 0$ depending only on the tiling, such that all tiles are enclosed by a disc of radius U.
We can now give the following definition:

Definition 4 (Tiling) *A tiling is a locally finite countable collection $\mathcal{T}$ of tiles $\{ T_1, T_2, ... \}$ such that:*

1. *Every tile is a polygon topologically equivalent to a closed disk*
2. *Every point in the plane is contained in at least one tile*
3. *The interiors of the tiles are pairwise disjoint*
4. *The tiles are uniformly bounded*

In addition to narrowing the type of shapes that form a tiling, we may focus on how the various tiles are arranged together and give some other restriction in this context. First of all as a singole shape has its own edges and vertices, we can think of the edges and vertices of a tiling. The union of the boundaries of all the tiles is called the *frontier* of a tiling and can be decomposed into a collection of *tiling vertices* and *tiling edges*. Tiling vertices are points that lie on the boundaries of three or more tiles. While tiling edges are curves (excluding their endpoints) that begin and end at tiling vertices and belong to exactly two tiles.
In a polygonal tiling we can distinguish the tiling vertices and edges from the vertices and edges of individual polygons. We call the latter *shape vertices* and *shape edges*. Shape vertices and edges are properties of tiles while tiling vertices and edges are topological properties of the assembled tiling. When shape vertices and the shape

edges coincide with the vertices and the edges of the tiling, then the tiling is called *edge-to-edge*.

Sometimes, it may happen that the intersection of two tiles consist of a collection of disconnected curves and points. We want to avoid these types of tilings because it makes hard to talk about a tiling edge shared by two tiles. So here we consider only tilings where the intersection of every two tiles is a connected set, that are usually called *normal tilings*.

If we consider a normal tiling $\mathcal{T}$, we can denote with V and E the tiling vertices and edges respectively and consider the set $\mathcal{T} \cup V \cup E$ to describe precisely the tiling. We can also enrich this set with a binary incidence relation between pairs of elements in $\mathcal{T} \cup V \cup E$ where two elements are related if they have a non-empty intersection. The set $\mathcal{T} \cup V \cup E$ together with this incidence relation forms the *combinatorial structure* of $\mathcal{T}$ which can be thought of as an infinite graph that records all adjacencies between tiling features. We can then say that a tiling $\mathcal{T}$ is *combinatorially equivalent* to a second tiling $\mathcal{T}'$ with vertices V' and edges E' if there is a bijection between the combinatorial structures of $\mathcal{T}$ and $\mathcal{T}'$ that maps vertices, edges and tiles to vertices, edges and tiles, and preserves the incidence relation.

Some tiling may require just a singole shape to be created, like those made up with hexagons, squares or equilateral triangles, instead others may require more than one type of tile, for example using octagons and squares or many others.

The minimal set of tiles $\mathcal{P}$ required to assemblate a tiling $\mathcal{T}$ up to isometries, is called *the prototile set* of $\mathcal{T}$ and we say that $\mathcal{P}$ *admits* the tiling $\mathcal{T}$. If $\mathcal{P}$ contains only one element then its admissible tilings are called *monohedrals*, instead if it contains k different prototiles then its tilings are called *k-hedrals*. Given a prototile set $\mathcal{P}$, we can create its admissible tilings by applying appropriate isometries to copies of its elements. For example we may use translations, rotations and sometimes even reflections if needed.

For this reason it becomes important to study the group of isometries of a tiling, usually called group of symmetries.

Definition 5 (Symmetry Group) *A symmetry of a tiling $\mathcal{T}$ is an isometry which*

maps every tile of $\mathcal{T}$ into a tile of $\mathcal{T}$.
The symmetry group $\mathcal{G}_{\mathcal{T}}$ of a tiling $\mathcal{T}$, is the group consisting of all the symmetries of $\mathcal{T}$.

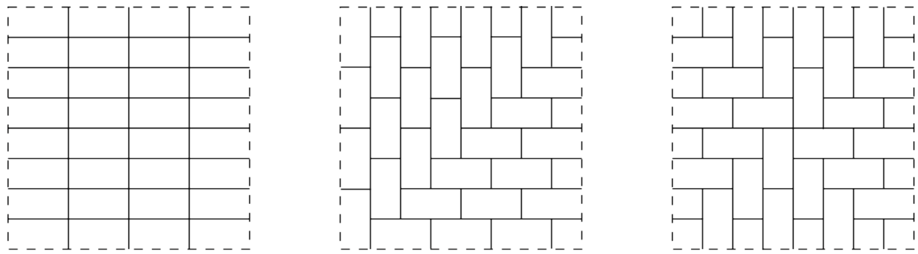
An informal and easy way to think of a symmetry of a tiling is the following. Imagine we have drawn the tiling on an infinite piece of paper and then traced it onto a trasparent sheet. A symmetry corresponds to a motion of the latter such that, after the motion, the tracing fits exactly over the original drawing.
If a tiling admits any symmetry in addition to the identity symmetry then it is called *symmetric*. The translations play a fundamental role among symmetries. We can give the following definition of periodicity.

Definition 6 (Periodicity) *A tiling $\mathcal{T}$ is:*

- *2-periodic, or even just periodic, if its symmetry group contains at least translations in two different directions,*
- *1-periodic, if its symmetry group contains translations in only one direction,*
- *non-periodic, if its symmetry group contains no traslations.*

Note that given a set of prototiles, they may tile the plane in many different ways, so with a set $\mathcal{P}$ there can be assembled more than one tiling $\mathcal{T}$. Some of these tilings could be periodic and others non-periodic.
There is nothing special in tiling the plane non-periodically, even a 2-by-1 rectangle can tile the plane non-periodically as shown in figure 2.2. More interesting are those shapes which admits tilings of the plane, yet no such tiling is periodic.

Definition 7 (Aperiodic prototile set) *A prototile set $\mathcal{P}$ is aperiodic if all its tilings of the plane are non-periodic.*



a b c

Figure 2.2: The figure shows three different tessellations made up with a 2-by-1 rectangle. In a) the tiling is 2-periodic with two translation simmetries one vertical and the other horizontal. In b) the tiling is 1-periodic with 45 degrees diagonal translation. Instead in c) the tiling is non-periodic since there are no translations, but it presents rotational symmetry.

2.2 Tiling by Regular Polygons

In this section we will study tilings by regular polygons, where each polygon has sides of the same length. If such a polygon has n sides and therefore n corners, we shall call it a *regular n-gon* and denote it by the symbol $\{n\}$. Thus $\{3\}$ is an equilateral triangle, $\{4\}$ is a square, and so on. Up to similarities, we assume that the sides of regular polygons have unit length.

If we want to assemblate a tiling by polygons, a fundamental condition that our prototiles must satisfy is that the sum of all the angles of the polygons that meet at each tiling vertex must adds up to 2π radians in order to fill in the space without gaps or overlaps. Recall that for a regular n-gon each angle has the same width and it is equal to: $(n-2)\pi/n$ radiands.

Let us first consider the simplest case of *regular tilings*, in which the prototile set $\mathcal{P}$ consists of a single element, that is, all the tiles are congruent to the same regular n-gon with $n \geq 3$. In this case around any tiling vertex we have a certain number $k \geq 3$ of adjacent tiles. So for the fundamental condition described before we get:

$$\frac{k(n-2)\pi}{n} = 2\pi \tag{2.1}$$

or equivalently:

$$(k-2)(n-2) = 4 \tag{2.2}$$

From the second version of this integer equation it's clear that the factors $(k-2)$ and $(n-2)$ must dived 4. So the only possibilities for these factors are: (1,4), (2,2) and (4,1). Which give the following pairs as possible results:

$$\begin{cases} k=3 \\ n=6 \end{cases}, \begin{cases} k=4 \\ n=4 \end{cases}, \begin{cases} k=6 \\ n=3 \end{cases}$$

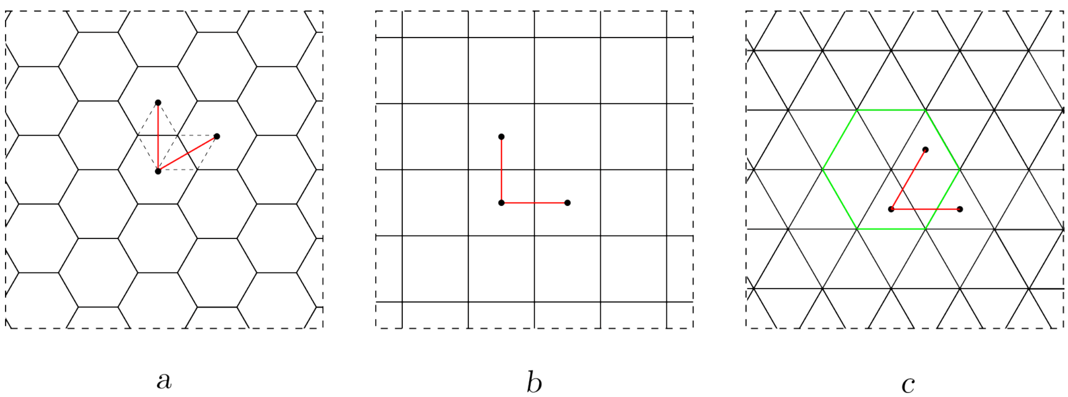


Figure 2.3: These are the only regular tilings of the Euclidean plane. The red lines indicate two independent translation vectors of the tilings. Instead in c) the green lines indicate how it is possible to divide the tiling into regular hexagons composed of 6 equilateral triangles.

meaning that the only regular tilings of the Euclidean plane are the familiar ones by regular hexagons, squares and equilateral triangles that are shown in figure 2.3.

We can observe that all three regular tilings are periodic. Those consisting of hexagons and squares present two obvious translational symmetries. In both cases we can reconstruct the entire tessellation by applying all the integer linear combinations of these two translations to a given tile. In the case of triangles, the situation is slightly different, we can in fact notice that in the tessellation, the triangles appear in two ways. There is a set of triangles with a vertex pointing upwards and a set with

a vertex pointing downwards. Clearly there is no translation that brings a triangle of the first set into a triangle of the second set, but there exist two different translations which can be applied to both sets of triangles that make the tasselletion periodic.
Another way to proceed, shown in figure 2.3, is to divide the tiling into groups of 6 triangles each of which forms an hexagon. So its periodicity follows from the periodicity of the hexagons. Alternatively, we could consider the parallelograms formed by the two different types of adjacent triangles. In this case, periodicity would follow from the fact that the parallelograms are affinely equivalent to the squares. That is, there is a unique affinity that transforms the square into a given parallelogram. Applying this affinity to the tiling by squares, we would obtain a tiling composed of parallelograms that are all congruent to each other.
This way of proceeding by dividing the tiling into groups of tiles to which it is possible to apply the two independent translations to realize the tiling, suggests the following definition of *fundamental domain.*

Definition 8 (Fundamental domain) *A fundamental domain for a tiling $\mathcal{T}$ is a finite connected union of tiles with which it is possible to reconstruct the entire tiling by applying the translations of its symmetry group to it.*

The fundamental domain exists only in the case where the tiling is periodic.
Let's now consider the *semi-regular tilings*, in which the prototile set $\mathcal{P}$ may contains more than one regular n-gons with $n \geq 3$. We assume that the tiling is *uniform*, that is, that all tiling vertices are equivalent to each other, in the sense that all of them are given by the intersection of the same types of tiles and each two vertices are related by some symmetry of the tiling. We use the notation $(n_1, n_2, ..., n_k)$ to indicate the sequence of the numbers of sides of the tiles around each vertex, where $n_i \geq 3$ and $k \geq 3$. This sequence is the same for all vertices up to cyclic permutations and inversion, due to the uniformity hypothesis.
As before, using the fundamental condition that must be satisfied by all tiling vertices, we can deduce the following equation:

$$\frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + ... + \frac{n_k - 2}{n_k} = 2 \tag{2.3}$$

or equivalently:

$$\frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k} = \frac{k-2}{2} \tag{2.4}$$

From 2.3 each addend of the first member is at least 1/3, so we have that $k \leq 6$. By studying in detail, case by case, all the possibilities for $3 \leq k \leq 6$, we obtain the following solutions for the sequence $(n_1, n_2, ..., n_k)$, up to cyclic permutations and inversion:

$$\begin{array}{ll}
(3,7,42)\ ,\ (3,8,24)\ ,\ (3,9,18)\ ,\ (3,10,15)\ ,\ (3,12,12) & \text{for } k=3; \\
(4,5,20)\ ,\ (4,6,12)\ ,\ (4,8,8)\ ,\ (5,5,10)\ ,\ (6,6,6) & \\
 & \\
(3,3,4,12)\ ,\ (3,3,6,6)\ ,\ (3,4,4,6)\ ,\ (4,4,4,4) & \text{for } k=4; \\
(3,4,3,12)\ ,\ (3,6,3,6)\ ,\ (3,4,6,4) & \\
 & \\
(3,3,3,3,6)\ ,\ (3,3,3,4,4)\ ,\ (3,3,4,3,4) & \text{for } k=5; \\
 & \\
(3,3,3,3,3,3) & \text{for } k=6;
\end{array}$$

We can observe that for case $k = 3$, all the solutions appears only one time because any permutation of the indices would produce the same tiling. For example, if we consider the solution $(4, 6, 12)$, around each tiling vertices we always find a 4-gon, a 6-gon and a 12-gon. Regardless of the order in which we arrange the polygons around each vertex, they will always produce the same tiling up to isometries. Instead for bigger values of k, the complexity increases. For $k = 4$ and $k = 5$, we have some solutions that are permutation of certain sequences, like $(3, 3, 4, 12)$, $(3, 4, 3, 12)$ and $(3, 3, 3, 4, 4)$, $(3, 3, 4, 3, 4)$. Here, as we can see from figure 2.4, the arrangement of the tiles is different and would not give rise to the same tiling, not even up to isometries. Although these 21 sequences are all the algebraic solutions of the equation 2.3, only 11 of them realize true tilings. One can check that the sequences $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(4, 5, 20)$, $(5, 5, 10)$, $(3, 3, 4, 12)$, $(3, 3, 6, 6)$, $(3, 4, 4, 6)$ and $(3, 4, 3, 12)$ don't realize a tiling because as you progress in building the tessellation, some tiles will overlap. We can also notice that the sequences $(6, 6, 6), (4, 4, 4, 4)$ and $(3, 3, 3, 3, 3, 3)$ are the regular tilings, so there are

only 8 new cases.
In figure 2.5 are shown these new tessellation found, which are all periodic like the regular ones. The black lines indicate two independent translations of their symmetry groups.

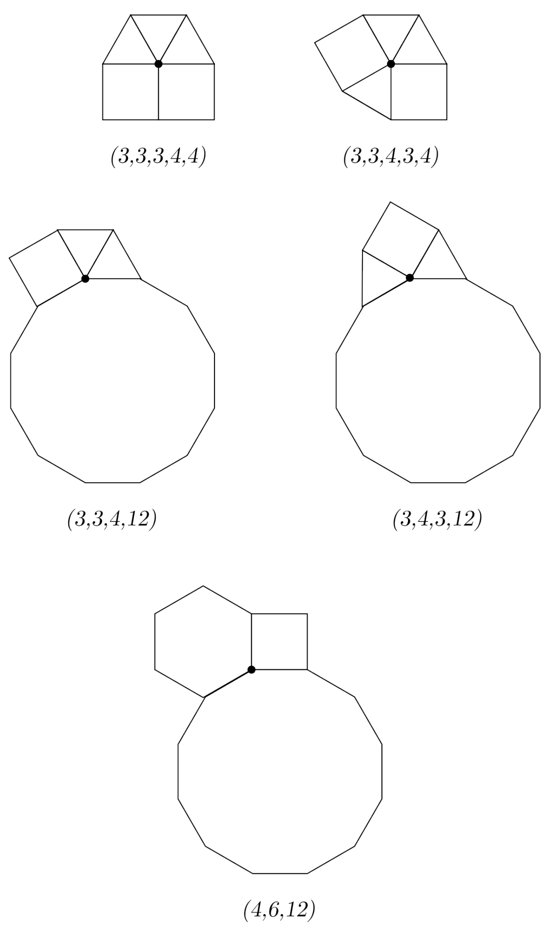
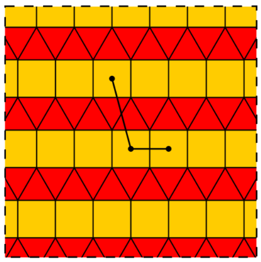
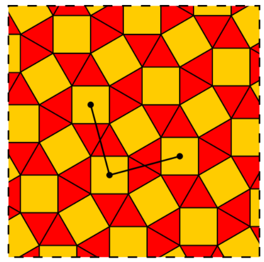


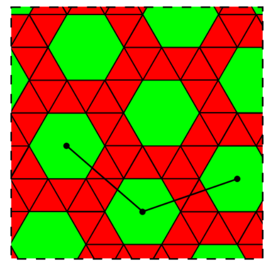
Figure 2.4: The first four images shows some permutations of the indices of the sequence $(n_1, n_2, ..., n_k)$ that do not produce the same tessellation up to isometries, for $k = 5$ and $k = 4$. Instead for $k = 3$ any permutation produces the same tessellation as shown in the last image.



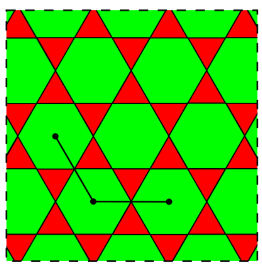
$T_{3,3,3,4,4}$



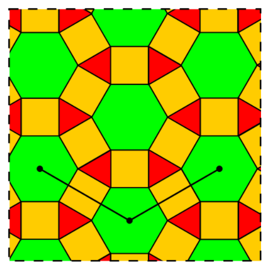
$T_{3,3,4,3,4}$



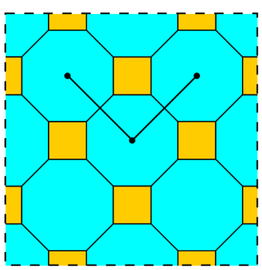
$T_{3,3,3,3,6}$



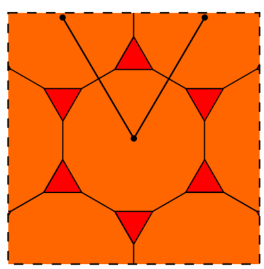
$T_{3,6,3,6}$



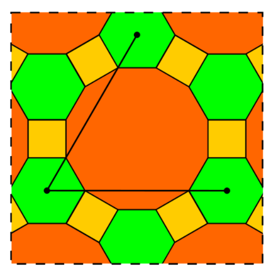
$T_{3,4,6,4}$



$T_{4,8,8}$



$T_{3,12,12}$



$T_{4,6,12}$

Figure 2.5

Lastly we introduce tilings with regular vertices, usually called *Laves tilings* in honor of the famous crystallographer Fritz Laves who studied these particular tilings. A tiling vertex is called regular if each consecutive pair of edges that meet in it form congruent angles. So if v edges meet at a regular vertex, they must form v angles of measure $\frac{2\pi}{v}$ each, with $v \geq 3$. Let's consider only the case of monohedral tilings with regular vertices. Our protile is an r-gon, not necessarily regular, with r regular vetices. Suppose that $v_1, v_2, ..., v_r$ are the numbers of edges that meet in each of the vertices of any tile. Since the sum of all interior angles of an r-gon is $(r-2)\pi$, then it holds:

$$\frac{2\pi}{v_1} + \frac{2\pi}{v_2} + ... + \frac{2\pi}{v_r} = (r-2)\pi \tag{2.5}$$

or equivalently:

$$\frac{v_1 - 2}{v_1} + \frac{v_2 - 2}{v_2} + ... + \frac{v_r - 2}{v_r} = 2 \tag{2.6}$$

which is the same equation as 2.3, with v_i instead of n_i. Thus the algebraic solutions are the same. However if one try to construct monohedral tilings with these solutions, only 11 of them form true tilings and the sequences correspond to the same found in the semiregular case. We denote these solutions with the notation $[v_1, v_2, ..., v_r]$ (with square brackets).

The sequences $[3,3,3,3,6]$, $[3,3,4,3,4]$, $[3,6,3,6]$, $[3,4,6,4]$, $[3,12,12]$, $[4,6,12]$, $[4,8,8]$ and $[6,6,6]$ form a unique tiling up to similarities, instead $[3,3,3,4,4]$, $[4,4,4,4]$ and $[3,3,3,3,3,3]$ form infinitely many tilings, the first two depending on a real value associated with the length of a side of the prototile and the last one depending on two real values. For $[4,4,4,4]$ and $[3,3,3,3,3,3]$, we can choose the same type of tiles as the regular ones, while for $[3,3,3,4,4]$ we can consider the prototile as the union of half a square and half a regular hexagon. With this special choise, semiregular and Laves tiling become deeply related. Indeed there is a one-to-one correspondence between the tiles, edges and vertices of the first one and the vertices, edges and tiles of the second one, in such a way that inclusion is reversed. That is, if in the first tiling a tile contains a certain vertex, then in the second tiling the corresponding vertex is contained in the corresponding tile. Two tilings with this correspondence are called *duals*, and the periodicity of one induces the same type of periodicity in the other. Thus also the Laves tilings are all periodic.

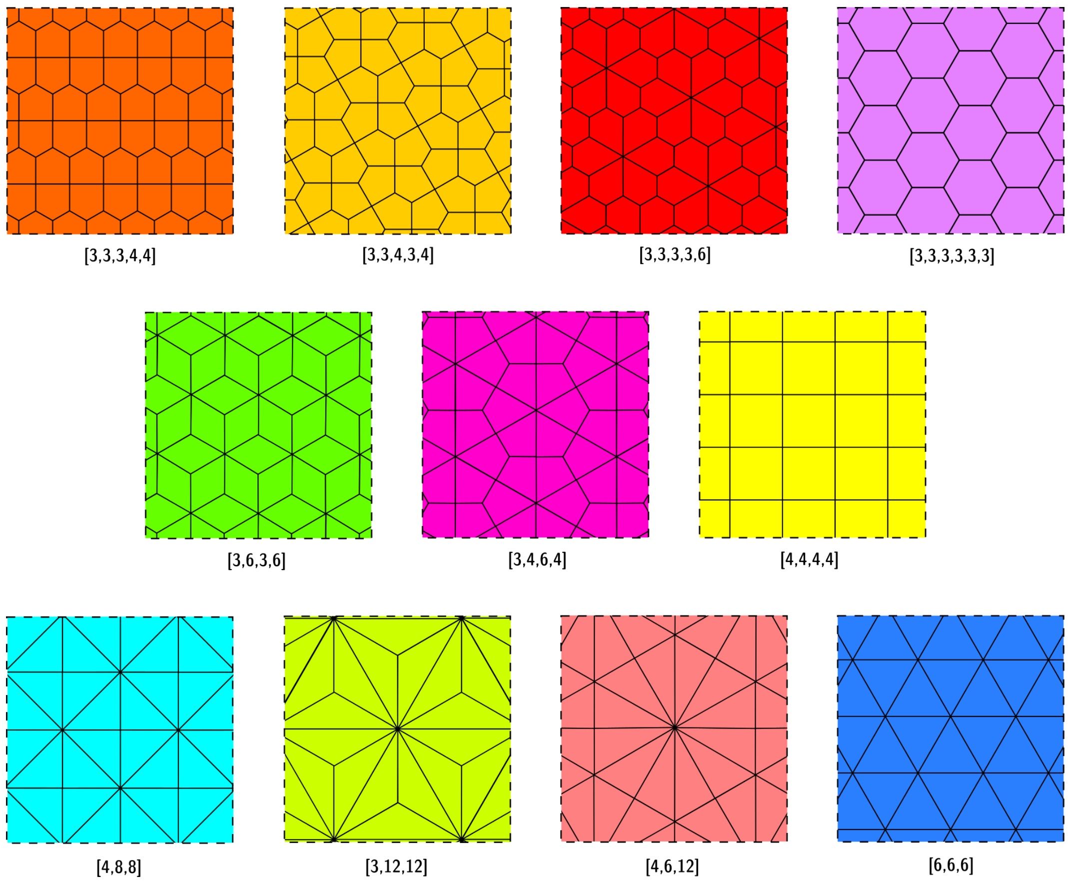


Figure 2.6

Furthermore if we superimpose these tilings, their edges intersect at right angles and the vertices of one correspond to the centers of the tiles of the other. In figure 2.6 are shown the 11 Laves tilings and in figure 2.7 there is a superimposition for $[3, 4, 6, 4]$.

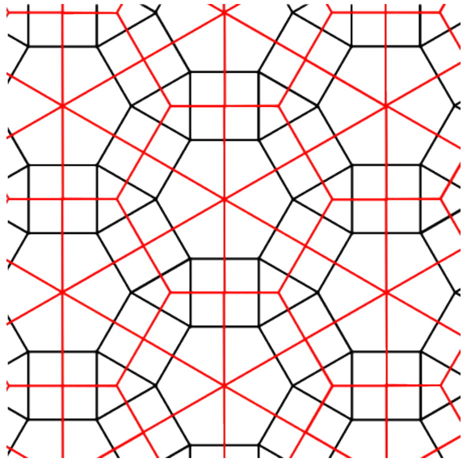


Figure 2.7: In the figure is shown a superimposition between the semiregular tessellation $(3, 4, 6, 4)$ in black and its dual Laves tiling in red.

2.3 Problem of existence of tilings

We now want to discuss the problem of the existence of tilings by a given prototile set, that is, we want to understand whether a given set of tiles is capable of tiling the plane. The problem is divided into two main cases, the periodic case and the non-periodic case. In the first case, we can exploit the fact that the symmetry group contains independent translations. Then it will be enough to focus on determining the fundamental domain of the tiling, which ensures that the tessellation exists and is periodic.

Instead in the non-periodic case, things get complicated and we cannot use the fundamental domain since it does not exist. So we need to introduce some new ideas and techniques.

First of all, we say that a tiling $\mathcal{T}_1$ is obtained by *composition* from a tiling $\mathcal{T}_2$, if each tile of $\mathcal{T}_1$ is the union of tiles of $\mathcal{T}_2$. Thus each edge of $\mathcal{T}_1$ will be a union of edges of $\mathcal{T}_2$, and each vertex of $\mathcal{T}_1$ will be a vertex of $\mathcal{T}_2$.

Definition 9 (k-composition) *Given two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$, if every tile of $\mathcal{T}_1$ is a union of k tiles of $\mathcal{T}_2$, then $\mathcal{T}_1$ is a k-composition of $\mathcal{T}_2$.*

We also say that two tilings are *similar* to each other, if one can be mapped onto the other by a similarity. It is possible for two monohedral tilings to be similar to compositions of each other. In particular a tiling is called a *similarity tiling* if it is similar to a composition of itself.

Definition 10 (k-similarity) *A monohedral tiling $\mathcal{T}$ is a k-similarity tiling if $\mathcal{T}$ is similar to a k-composition of itself, and no smaller value of $k > 1$ has this property.*

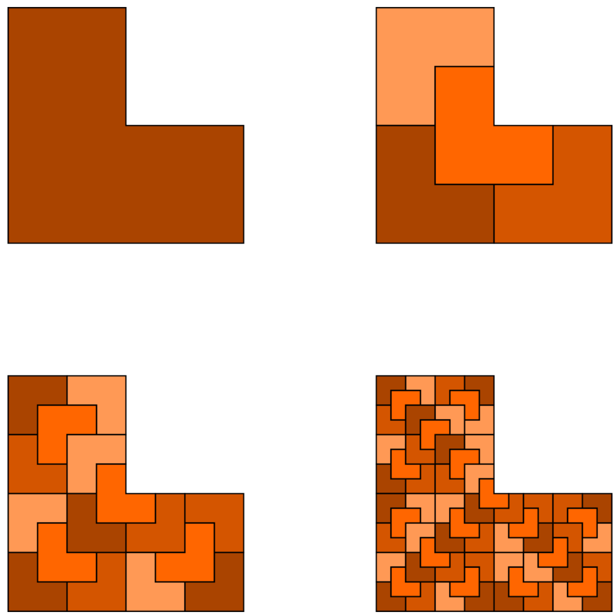


Figure 2.8: A 4-similarity tiling called "the chair".

For some k-similarity tilings there are many different ways in which unions of sets of k tiles of $\mathcal{T}$ can be taken as the tiles of a tiling similar to $\mathcal{T}$. In other cases this composition process is unique. This would not be valid if k were not uniquely defined by minimality.

We can now prove the following result:

Lemma 1 *If the k-composition process for a monohedral k-similarity tiling is unique, then $\mathcal{T}$ is not periodic.*

Proof.

We can show that $\mathcal{T}$ cannot possess a translation as a symmetry. Suppose that there were such a translation τ_v by a vector v of length d. Then uniqueness of composition implies that τ_v must also be a symmetry of the k-composed tiling. Applying this argument repeatedly to further k-compositions we obtain tilings with arbitrarily large tiles. But it is impossible for τ_v to be a symmetry of any tiling in which every tile contains a circular diameter larger than d, and hence we arrive to a contradiction which proves the statement. In other words these tiles would have internal points in common with their own translated, which is impossible. $\square$

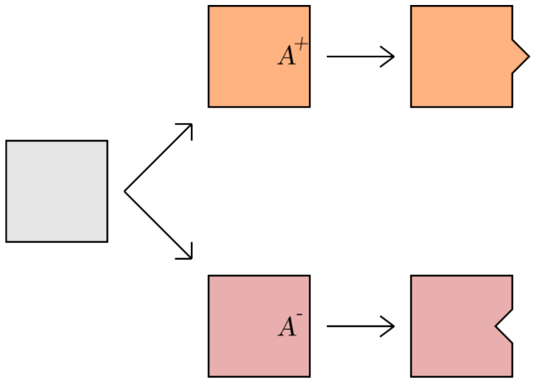
Figure 2.8 shows an example of a 4-similarity tilings, called the chair. We can observe that, using four tiles, it's possible to construct a bigger tile that is similar to the original one. In particular, in the last image we can see that it's possible to apply the 4-composition process to the entire tiling in a unique way. So it follows that the tiling is non-periodic.
Regarding the above Lemma, notice that a tiling $\mathcal{T}$ is given by the hypothesis. But how can one prove that a set of prototiles can actually tiles the whole plane?
To Answer this question we need to prove the so called *Extension Theorem*.

Theorem 4 (Extension Theorem) *Let $\mathcal{P}$ be a finite set of polygonal prototiles. If there exist an edge-to-edge tiling by $\mathcal{P}$ of regions containing arbitrarily large disks, then there exists at least one edge-to-edge tiling by $\mathcal{P}$ of the entire plane.*

Proof.
Up to translations we can assume that the disks in the statement are all centered on the origin. We can then construct a sequence of tilings $\mathcal{T}_n$ with $n \geq 1$, such that each $\mathcal{T}_n$ covers a connected region containing the disk D_n centered on the origin of radius n. For each $n \geq 1$ let $T_{0,n}$ be a tile of $\mathcal{T}_n$ containing the origin. Since the set $\mathcal{P}$ is finite, we can also assume, passing to a subsequence, that the tiles $T_{0,n}$ are all congruent to the same prototile. So up to congruences we can also assume that these all coincide with the same tile T_0. At this point we proceed with a diagonal argument as Cantor. Given an edge l of T_0, there is only a finite number of possible

tiles adjacent to T_0 along l, so there must exist a subsequence of tilings in which such tiles are always the same, say T_1.
Repeating this process with any other boundary edge l of $T_0 \cup T_1$, we find a further subsequence of tilings in which there is always the same tile T_2 sharing side l with $T_0 \cup T_1$. And so on... Assuming that we have already identified the tiles $T_0, ..., T_{n-1}$, we repeat this process with a side l of the edge of $T_0 \cup ... \cup T_{n-1}$ to identify the next tile T_n adjacent to the previous ones along the edge l in all the tilings of a suitable subsequence. This gives a tiling $\mathcal{T} = \{ T_n \}$ with $n \geq 1$ of the entire plane. The only precaution, so that no gaps remain, is to consider the edges l so as to first exhaust those closest to the origin. $\square$

Often it is usefull to add some marking or decoration to the tiles in order to impose some restrictions to the way we can put tiles together. This method gives us the so called *matching conditions* that tell us how to put the tiles together. For example we can impose matching conditions to simple tiles such as squares to produce non-periodic tessellations. These conditions can be decoded by modifying the shape of the tiles. An example is shown in the figure 2.9.



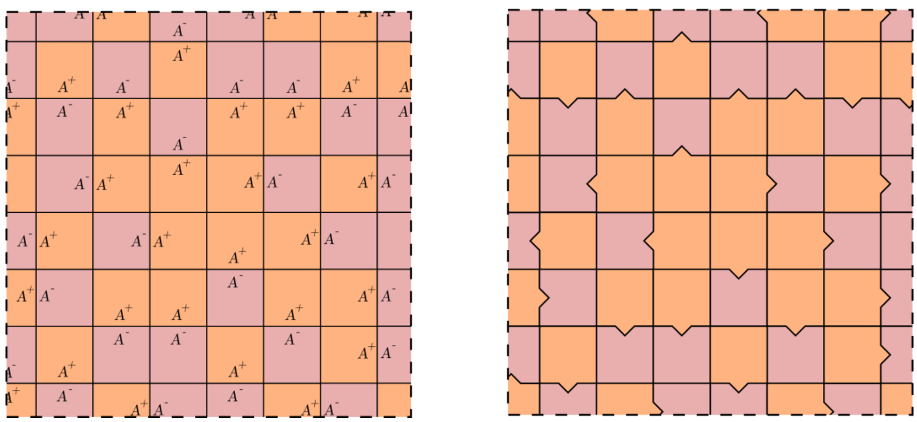


Figure 2.9: Above we can see an example of a simple matching condition applied to a square. The square with the side marked A^+ must be put next to the square marked A^- so that the two marks fit together. Below we can see how it is possible to create a non-periodic tiling using only squares with matching conditions. On the left we have the tiling by marked squares while on the right we have used the squares with modified sides that respect the matching conditions.

Chapter 3

Penrose aperiodic tiles

In the early 1960s, the mathematician Hao Wang conjectured that can't exist a finite aperiodic set of tiles. If this were true then the tiling problem, that is establishing whether with a given set of prototiles one can tile the plane or not would have been decidable. So there would have been an algorithm capable of solving the problem in a finite time for any set of prototiles. Instead in 1966 Robert Berger, a student of Wang, managed to construct an aperiodic prototile set, thus showing that the tiling problem is undecidable. His set consisted of 20426 tiles, obtained by decorating a single square in order to impose appropriate matching conditions. Later, in 1971 Raphael M. Robinson found an aperiodic set of 6 tiles and other sets were discovered in the late 1970s by Robert Ammann.
The smallest aperiodic sets, until 2023, were discovered by Roger Penrose in 1973 and 1974. He proposed three sets of prototiles, called P_1, P_2 and P_3, the first consisted of 6 shapes, while the other two consisted only of 2 shapes. In this chapter we will focus on P_2, showing that, it can only tile the plane non-periodically.

3.1 Kite and Dart

The aperiodic prototile set P_2, is made up of two tiles, called the *Kite* and the *Dart*, which for simplicity we denote with K and D.
They both are 4-gons with two sides of length 1 and two sides of length Φ (the golden

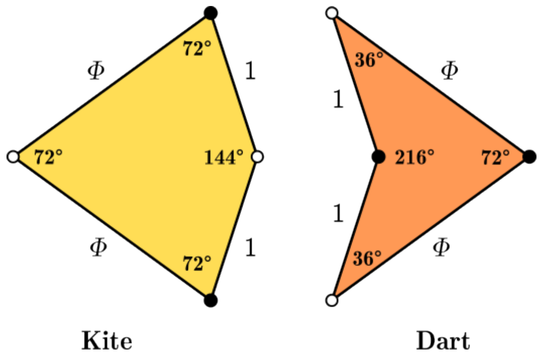


Figure 3.1

ratio $\Phi = \frac{\sqrt{5}+1}{2}$). The Kite is a convex polygon, with interior angles of $72, 72, 72, 144$ degrees, while the Dart is concave polygon with $36, 216, 36, 72$ degrees angles. They are show in figure 3.1. Since all the angles are integer multiples of 36 degrees, or equivalently $\pi/10$ radians, it's convenient to refer to them as multiples of $\alpha = \pi/10$. It's important to notice, that there are black and white dots at the vertices. These colored dots impose a matching condition. A black vertex of a tile must touch a black vertex of another tile, and similarly, white vertices must match together. Without these markings, the tiles K and D could give rise to periodic tilings. For example, is possible to form a romb by attaching the Kite to the Dart so that the 144 degree angle touches the 216 degree angle.

Our goal is to show that the set P_2 can only tile the plane non periodically with the given matching conditions.

We can start by noticing that there are only 7 ways to arrange the tiles around a vertex, they are shown in figure 3.2. The mathematician John Conway called these 7 unique sets of tiles with the names: Sun, Star, Jack, Ace, Deuce, Queen and King. Any other combinations of Kites and Darts around a vertex it's impossible, because the tiles either would overlap, or not meet the matching conditions.

Here we will follow the technique used by Robinson to investigate the properties of these tiles. The first step consists on subdividing K and D along their axis of symmetry indicated with the dashed lines in figure 3.3. By doing this we get four isosceles triangles, that we denote with the letters A_1, A_2, B_1 and B_2. The A_s come

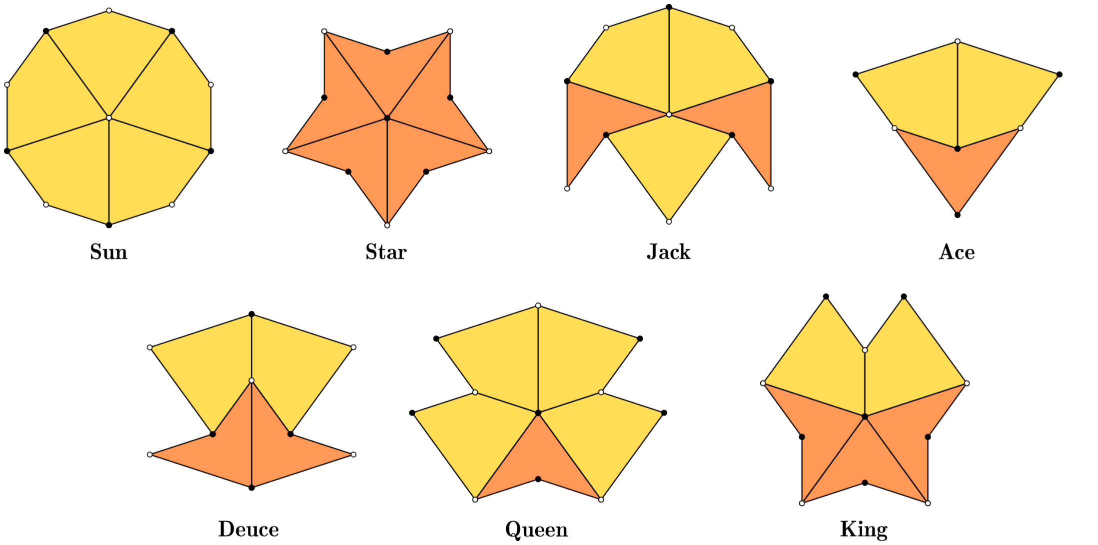


Figure 3.2: These are the only 7 ways to arrange the Kites and the Darts around a tiling vertex.

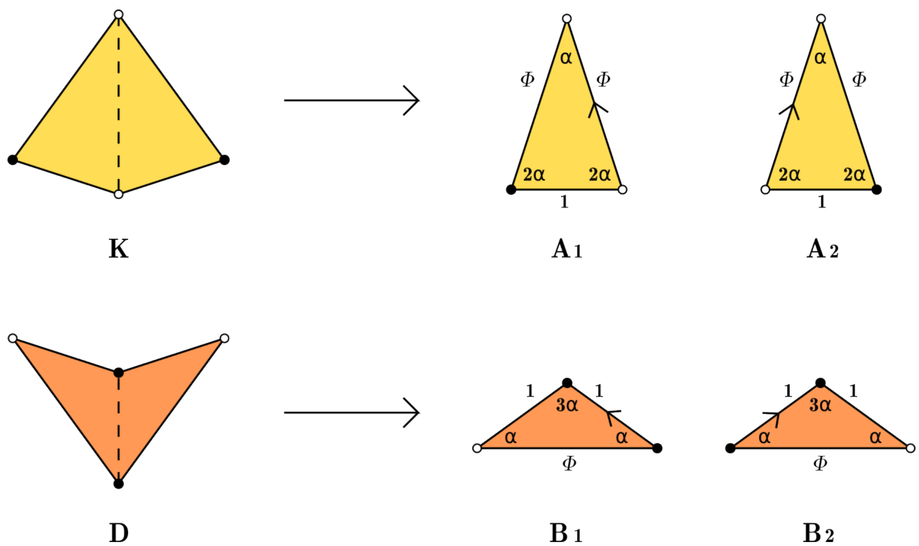


Figure 3.3

from the Kite, while the B_s come from the Dart, moreover they are two pairs of congruent triangles that differ on the vertex markings. If we apply a reflection to A_1, we get A_2, and the same it's true for B_1 and B_2.

Suppose that we have a tiling by K and D, then there is a unique way to get a tiling by A_s and B_s by subdividing as shown. However, we need a unique way to go from a tiling by A_s, B_s to a tiling by K and D. In order to achieve this purpose, we need to impose an additional matching condition to the A_s and B_s tiles. We can draw an arrow pointing to the peak of the isosceles triangle on the edge which has equal vertex colours, and require that the arrow directions must match when two tiles share an edge with arrows. In this way we avoid that, for example, two A_s may share the edge having white vertex colours in the opposite way as that which comes from subdiving K. So by removing the edges with equal vertex colours, there is a unique way to go from tilings by A_s and B_s to tilings by K, and D.

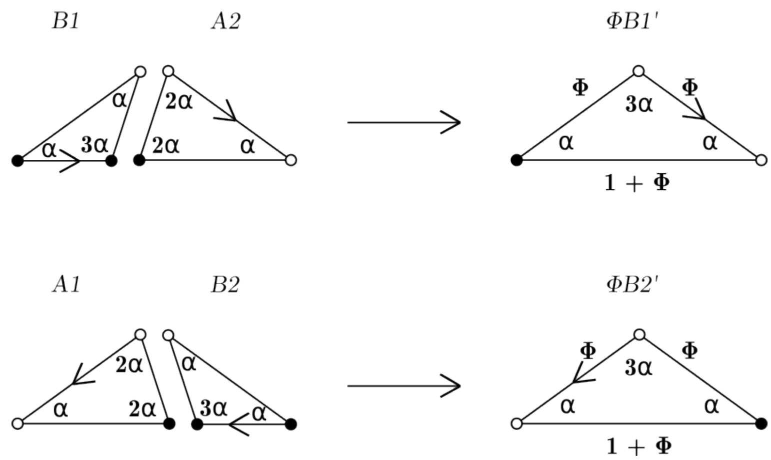


Figure 3.4

The second step consists on creating two new tiles, that we refer to as $\Phi B_1'$ and $\Phi B_2'$, by composing the A_s with the B_s. To create $\Phi B_1'$, we attach B_1 to A_2 along the shortest edge of length 1, while for $\Phi B_2'$ we do the same thing using B_2 and A_1 as shown in figure 3.4. We can notice that the $\Phi B_s'$ are similar to the B_s, indeed they have the same angles, and the similarity ratio is Φ. Recall that the golden ratio satisfies the equation $\Phi^2 = \Phi + 1$. The only difference is that the colored dots at the vertices and the direction of the arrows are reversed. For this reason, we used the prime ($'$) notation which means that the dots and the arrows are inverted. We can also notice that the B_s tiles are always forced to be attached to an A_s tile along the shortest edge. Indeed if you try to attach two copies of the B_s along the short edge, either the vertex colours fail, or the angles around a vertex add up to more than 2π.

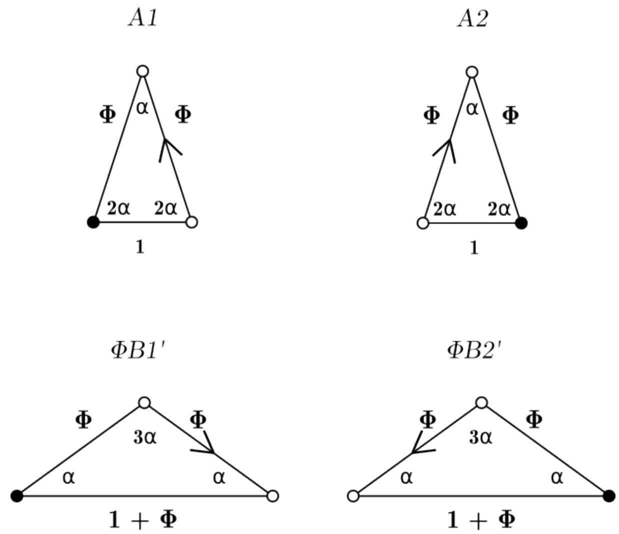


Figure 3.5

Figure 3.5 shows the new set of tiles. If we apply this composition rule, we get a tiling by A_s and $\Phi B_s'$ tiles. As before this composition process is unique. Moreover we can come back to an A_s, B_s tiling in a unique way, by subdividing all the $\Phi B_s'$ into two A_s and B_s tiles by reversing the composition process.

The third step consists on creating another couple of tiles, namely $\Phi A_1'$ and $\Phi A_2'$, by composing the A_s with the $\Phi B_s'$. To create $\Phi A_1'$, we attach A_1 to $\Phi B_1'$ along the edge with the arrow, while for $\Phi A_2'$ we do the same thing using A_2 and $\Phi B_2'$ as shown in figure 3.6. As before the $\Phi A_s'$ are similar to the A_s, and the dots at the vertices and the direction of the arrows are reversed. Furthermore, similarly as in step two, the A_s tiles are always forces to be attached to a $\Phi B_s'$ tile along the edge which has an arrow. And if you try to attach two copies of A_s along the edge with the arrow, the tiling fails, because, since only the A_s tiles have an edge of length 1, you would need four A_s tiles attached together. But then you can't put other tiles to fill in the gaps, because the vertex colours or the arrows fail, or the sum of the angles around a vertex add up to more than 2π.

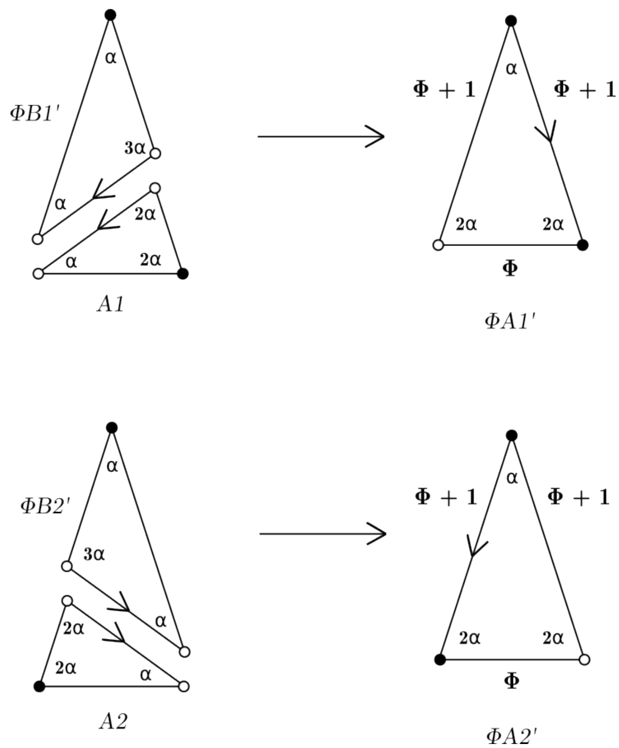


Figure 3.6

Figure 3.7 shows the new set of tiles. If we apply this composition rule, we get a tiling by $\Phi A'_s$ and $\Phi B'_s$ tiles. As before this composition process is unique. Moreover we can come back to an A_s, $\Phi B'_s$ tiling in a unique way, by subdividing all the $\Phi A'_s$ into two A_s and $\Phi B'_s$ tiles by reversing the composition process.

At the moment, applying steps 1,2 and 3 changes tile types as follows:

$$K,\, D \Rightarrow A_s,\, B_s \Rightarrow A_s,\, \Phi B'_s \Rightarrow \Phi A'_s,\, \Phi B'_s$$

The fourth step consists on switching the vertex colours and arrow directions. In this way we can get rid of the prime ($'$) notation, and we get the ΦA_s, ΦB_s that have the same vertex colours and arrows orientations as the original A_s, B_s tiles.

Finally, the fifth and last step consists on applying the reverse of step 1 to the ΦA_s, ΦB_s tiles. In practice, we erase the edges with arrows, ending up with larger version of the original Kites and Darts rescaled by a factor Φ, that we can refer to as ΦK,

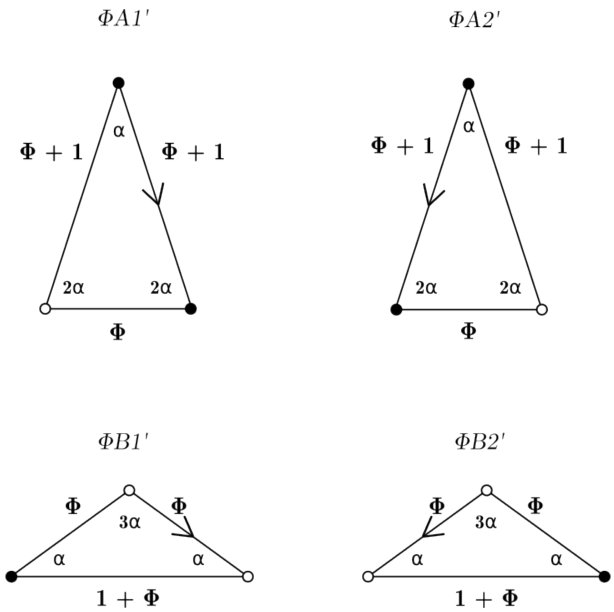


Figure 3.7

ΦD.

$$\Phi A'_s,\ \Phi B'_s \Rightarrow \Phi A_s,\ \Phi B_s \Rightarrow \Phi K,\ \Phi D$$

So from a tiling by K, D we can uniquely get a tiling by ΦK, ΦD by following steps 1 to 5. We can call this whole process *Composition*, in which we take unions of tiles to build up larger tiles, similar to the original ones and with the same matching conditions. Note that we can continue to apply the Composition process to the composed tiles as many times as we want, obtaining tilings by $(\Phi^2 K, \Phi^2 D)$, $(\Phi^3 K, \Phi^3 D)$ and so on.

Since each individual step of the Composition process is uniquely reversible, we can also performe all the steps in the reverse order. Starting from a tiling by K and D, we obtain a tiling by $\Phi^{-1}K$ and $\Phi^{-1}D$ tiles. We refer to this inverse process as *Decomposition*. In practical terms, both Composition and Decomposition involve simply drawing a new tiling over the existing one. During Composition, you draw larger tiles, scaled by a factor Φ, whereas during Decomposition, you draw smaller tiles, scaled by Φ^{-1}. In both cases, starting from a partial tiling that covers a region

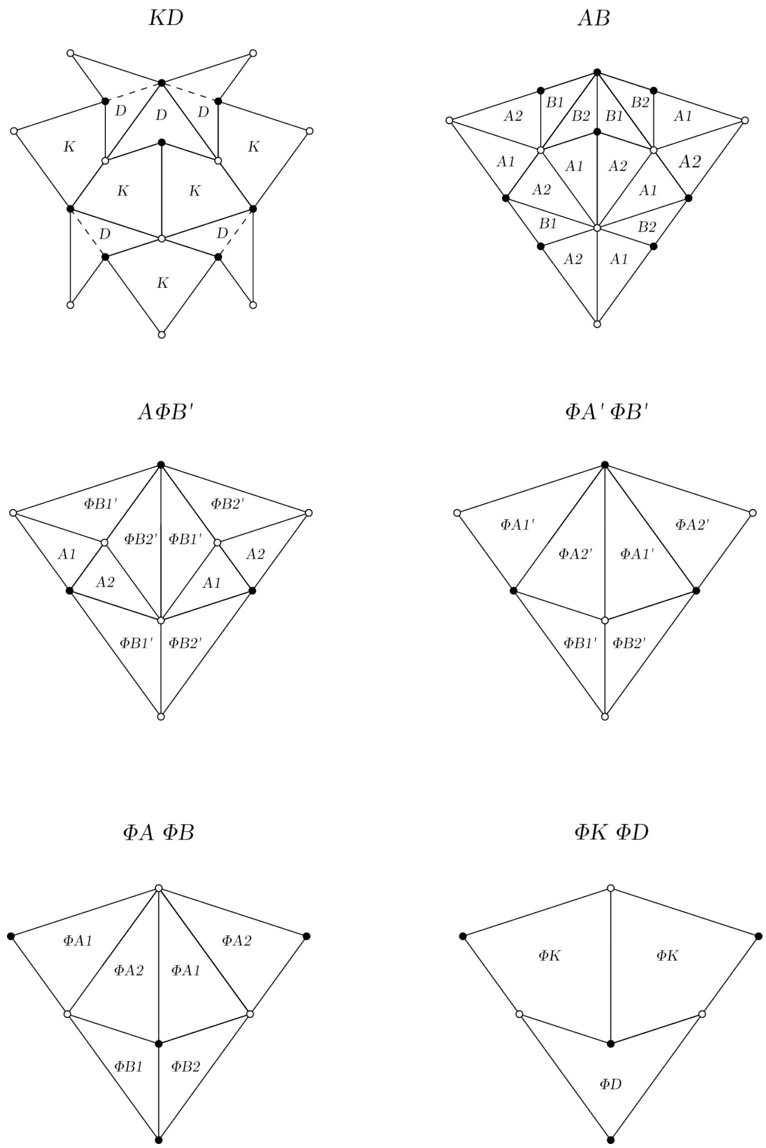


Figure 3.8: The figure shows the Composition process applied to a partial tiling by Kites and Darts. The dashed lines in the first picture indicate half darts that, for the purpose of demostration have to be removed in step 1. Otherwise going forward in the steps there would be some tiles that should not be present, such as the B_s together with the A_s and $\Phi B_s'$. Of course this problem only occurs in the case of partial tessellations.

R, the resulting composed or decomposed tilings will likewise cover approximately the same region R except for some boundary triangles.
Another fundamental procedure that we need to introduce, is the so-called *Inflation*. It consists in applying the Decomposition to a region covered by tiles and then rescale them by a factor Φ, in order to bring them back to their original size. Since, the Decomposition increases the number of tiles, when we rescale them, we obtain a partial tiling covering a region having approximately the same shape, but Φ times larger.
Using the Inflation we can finally show that the Kite and Dart tiles can actually tile the plane.
Take any partial tiling by K and D covering some small disc of radius r, for example one of the 7 unique sets shown at the beginning. If we now apply Inflation n times, we get a partial tiling by K, D covering a disc of radius $\Phi^n r$. Thus, we are able to produce partial tilings covering arbitrarily large discs. So from the Extension Theorem it follows that K and D tiles can tile the whole plane.
Lastly we have to show that all the tilings produced by K and D are non-periodic, we will use the same argument of uniqueness of composition used in Lemma 1.

Theorem 5 *K, D are an aperiodic prototile set.*

Proof.
Let $\mathcal{T}$ be a tiling by K, D and suppose by contradiction that there is a translation τ_v by a vector v of length d, such that $\tau_v(\mathcal{T}) = \mathcal{T}$. Now consider a patch of the tiling covering a disc of radius bigger than d. When you apply the translation to this first patch, you get a second patch. The two patches must agree on the overlap, since that translation is a symmetry of $\mathcal{T}$.
If we apply Composition to the first and the second patches, we obtain two other patches by ΦK, ΦD, that must agree on the overlap because the Composition is uniquely determined. As a result the two patches obtained after Composition have the same translational symmetry.
So if we apply Composition N times, we obtain a tiling by $\Phi^N K$, $\Phi^N D$ with the same translational symmetry. But for N such that $\Phi^N > d$, this is absurd, because

the translation will move a given tile to a new tile which overlaps with the given tile, but this contradicts that τ_v is a symmetry. In other words, these big tiles would have internal points in common with their own translated, which is impossible. $\square$

Chapter 4

Aperiodic monotile

In this last chapter, we will talk about a recent discovery in the field of aperiodic sets of tiles.

For decades it was believed that a single shape (an "einstein" from the german "ein stein", literally "one shape"), could not tile the plane only in a non-periodic way. Contrary to this belief, in March 2023, the team formed by David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss, brought to everyone's attention the first example of an aperiodic monotile, a single geometric shape capable of tiling the plane only in a non-periodic way, called by the team "the hat polykite".

We will report the proof published by the team in the paper "An aperiodic monotile". This proof makes use of a new technique, which consists on showing that in any tiling by the hat polykite, every tile belongs uniquely to one of four distinct clusters called by the authors *metatiles*. Then it proceed with a Berger-style inductive approach, showing that any tile in any tiling by these four metatiles lies in a unique hierarchy of *supertiles*, that are combinatorial copies of the metatiles at larger and larger scales. Thus the aperiodicity of the hat comes from the uniqueness of these hierarchies of supertiles.

Although the hat polykite is the first example of aperiodic monotile, it requires unreflected and reflected tiles in every tiling, leaving open the question of whether a single shape can tile aperiodically using only translations and rotations. Some months after the discovery of the hat, the same authors published the paper "A chiral aperiodic monotile" where they presented another shape called "the spectre", closely related to the hat, that has the ability to tile the plane without the use of reflected tiles. [7]

4.1 The hat polykite

David Smith, an amateur mathematician and tiling enthusiast, is the one who discovered the einstein. In November 2022, while experimenting with new types of shapes using the PolyForm Puzzle Solver software, he came up with a peculiar 13-gon. This shape seemed to have the ability to cover large regions of the plane, and no matter how hard he tried to find periodic patterns, he couldn't find any. He therefore contacted Craig S. Kaplan to inform him about his potential discovery, who in turn contacted Joseph Samuel Myers and Chaim Goodman-Strauss to help him complete the proof.

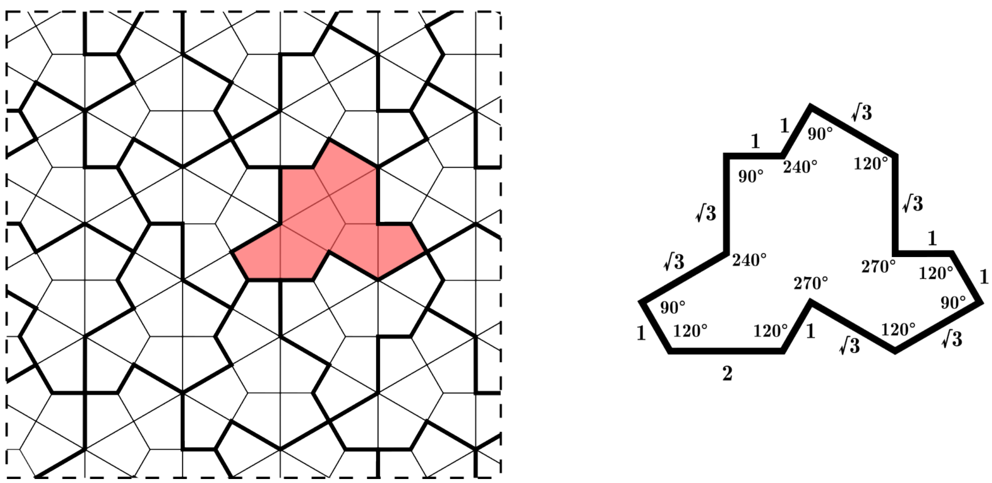


Figure 4.1

The hat polykite is a 13-gon, union of eight kites in the Laves tiling $[3,4,6,4]$ with no additional matching condition. Its complex shape it's enough to ensure that it is an aperiodic monotile. It has interior angles of $90, 120, 240$ and 270 degrees, with edges which come in parallel pairs of length 1 and $\sqrt{3}$. The edge of length 2 can be

thought of as two consecutive edges of length 1. Figure 4.1 shows a partial tiling by the hat polykite superimposed with the $[3, 4, 6, 4]$ Laves tiling and a single hat labeled with the measures of all its angles and edges. It's important to notice that the hat has no axis of symmetry, so the reflected tile is distinct from the unreflected. In order to study the behavior and the properties of the hat, the authours constructed a few large partial tessellations using a softwar. Of course this approch does not guarantee that the observed properties will remain valid in an infinite tiling, but it's certainly a starting point to begin to advance hypotheses that will then need to be verified. For example, they observed that the reflected tiles are always distributed sparsely and evenly within a set of unreflected tiles, and that every reflected tile is surrounded by three unreflected hats always arranged in the same way. They called this group of tiles "the shell".
In figure 4.2 we can see a partial tessellation by the hats, where the reflected tiles are colored dark blue, while the three unreflected tiles surrounding them are colored light blue. Using this coloring, the authors were able to identify two additional tile clusters for the remaining tiles that do not surround the reflected hats. The first one consists of a single hat, and it is always surrounded by three shells. The second one instead consists of two hats and its shape resembles that of a parallelogram. This parallelogram-shaped cluster appears in two varieties. It can be found adjacent to two shells, or it is attached to two other parallelograms, forming a three-armed propeller shape called by the authors, "the triskelion".
In figure 4.3 we can see the same partial tessellation as in figure 4.2 decomposed into these four clusters. This method of decomposing the tassellation is the starting point for the proof of aperiodicity of the hat polykite. In the next section we will see in detail how the authors proved that any tiling by the hat polykite can be divided into the clusters shown.

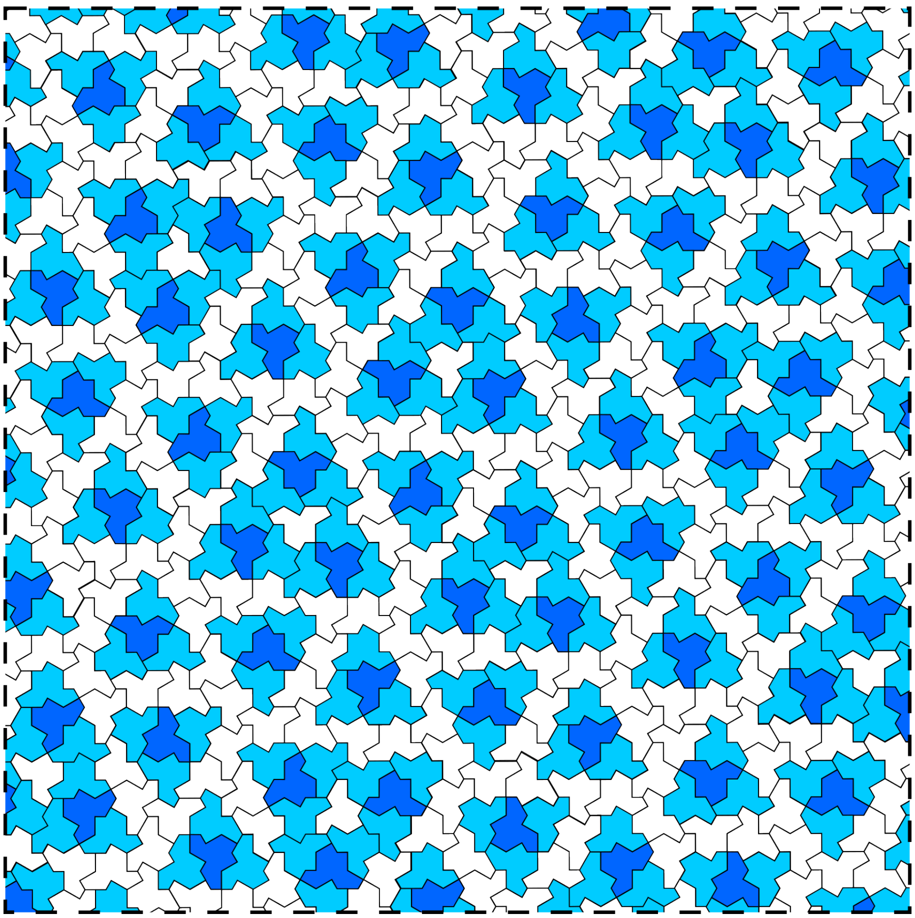


Figure 4.2: A partial tiling by the hat polykite

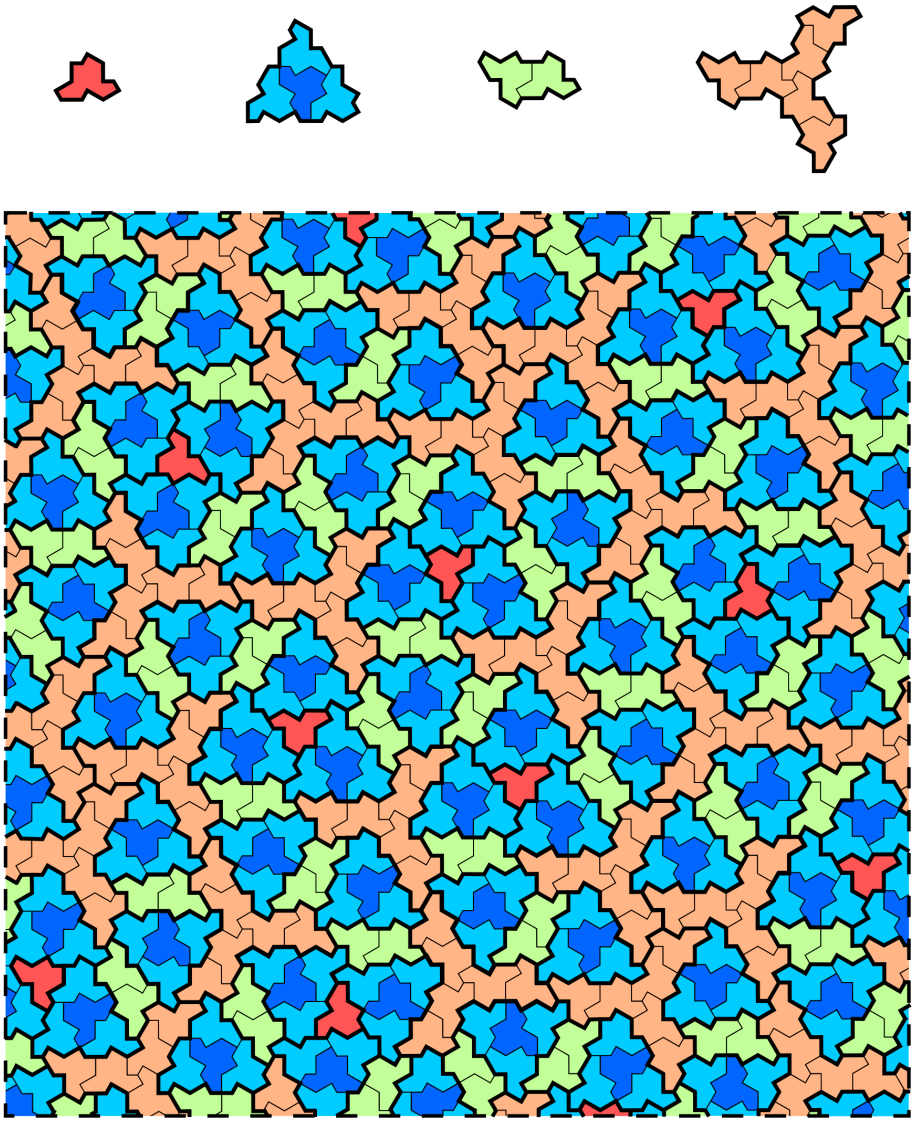


Figure 4.3: A partial tiling by the hat polykite decomposed into the four clusters of tiles. On top of the tiling are shown the clusters. In order: the single hat in red, the shell in light and dark blue, the parallelogram in green and the triskelion in orange.

4.2 Metatiles

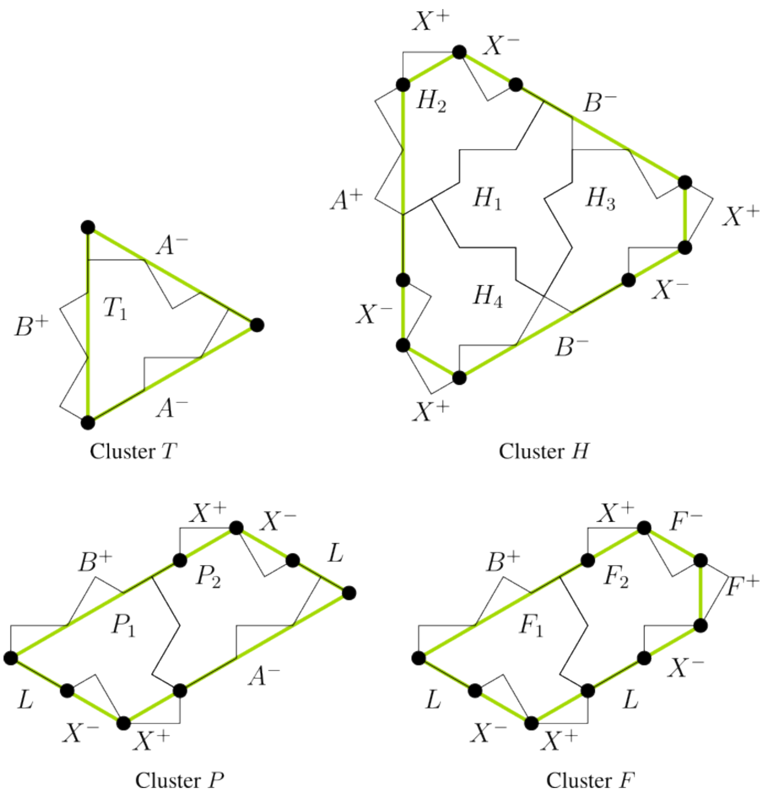


Figure 4.4: The four metatiles T, H, P and F.

Given the four clusters, we can define simplified tile shapes called metatiles, which we refer to as T, H, P and F. They are shown in figure 4.4. Each metatile is a polygon (outlined in lime) that approximates the shape of a cluster. The hats that make up a cluster have a unique label. Along the boundaries there are some triangles (half of a kite) that are not aligned with the edges of the metatiles. Some extend outwards from the edges, others inwards. We can use these irregularities to define matching conditions for the metatiles. Note that along the boundaries there are black dots that divide each edge in labelled segments. These labels indicate the matching conditions that ensure that any tiling by metatiles induces one by the hat polykite. When two metatiles meet along an edge, the segments marked A^+ must match with A^-, B^+

with B^-, X^+ with X^-, F^+ with F^- and L with L. Moreover the four metatiles must form a tiling using copies that are only rotated and not reflected. Since the matching conditions do not permit a reflected cluster to adjoin a non-reflected cluster, it then follows that either no clusters are reflected or all clusters are reflected. So without loss of generality we can assume that no clusters are reflected.

Our goal is to show that any tiling by the hat polykite can be divided into the clusters shown, satisfying the matching condition with the resulting tiling by metatiles having the same symmetries as the original tiling by polykites. To do so we need to introduce the notion of patchs of tiles.

Definition 11 (Patch of tiles) *A patch of tiles is a collection of non-overlapping tiles whose union is a topological disk.*

A 0*-patch is a patch containing a single tile, and an* $(n+1)$*-patch is a patch formed from the union of an* n*-patch* P *and a set* S *of additional tiles, such that* P *lies in the interior of the patch and no proper subset of* S *yields a patch with* P *in its interior.*

We can think of an n-patch as a tile surrounded by n concentric rings of tiles.

Since each metatile is small enough to be contained into a 2-patch, we can show that any tiling by the hat polykite can be divided into the four clusters satisfying the matching conditions, by a case analysis of all possible 2-patches of hats. First we need to define some rules to assign labels to the tiles in the patch's interior based only on its immediate neighbours. If our rules involve no arbitrary choices, then they preserve all the symmetries of the tiling. Then we must check that the labels are consistent with those of the clusters. Two things have to be verified. First, when the central tile of a patch has a given label from one of the clusters, its neighbours in that cluster must appear with the correct labels in the expected positions and orientations within the patch. Second, when a patch's central tile is adjacent to a tile with a label from a different cluster, their adajcency relationship must be consistent with the labelled edge segments that define the matching conditions for the clusters. If all the 2-patches, pass all the checks successfully, then we have done.

The first step of this proof consists on generating all the possible 2-patches that can appear in tilings by hats. Unfortunately it's quite difficult to create such an

exact list of all possible patches. But we can generate a bigger list that include false positives that do not occur in any tilings and every 2-patch that can occur, as long as the analysis produces valid results for them as well. The authors of the paper generated this list with a software and it consists of 188 surroundable 2-patches, that is 2-patches that can be surrounded at least once more to form a 3-patch. Another thing to know is that, it can be proved that any tiling by the hat polykite is aligned with the underlying $[3, 4, 6, 4]$ Laves tiling. So the analysis can be reduced to those patches aligned with the $[3, 4, 6, 4]$ tiling.
The next step of the proof is computer-assisted and it consists on assigning the labels to the tiles of each patches and performing all the checks to them.
We will present the *classification rules* for assigning labels and the two types of checks that need to be done: the *within-cluster matching checks* and the *between-cluster matching checks*. Then we will apply this rules and these checks to a surroundable 2-patch to see how they work in practice. Of course all the surroundable 2-patches passed all the checks successfully. This gives us the following result: [6]

Proposition 2 *Any tiling by the hat polykite can be divided into the clusters shown in Figure 4.4 (or reflections thereof, but not mixing reflected and non-reflected clusters), satisfying the given matching conditions, with the resulting tiling by metatiles having the same symmetries as the original tiling by polykites.*

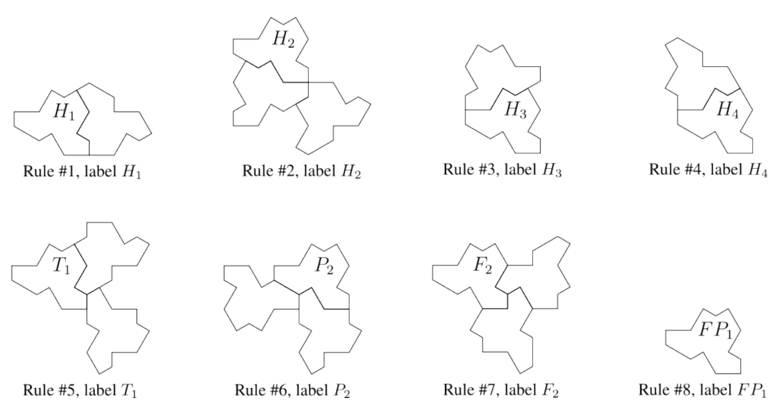


Figure 4.5: The eight classification rules for tiles.

In figure 4.5 are shown the eight classification rules for tiles. In each diagram there is a labelled tile and some of its neighbours. We will use these rules to assign labels to the central tile and its neighbours that form an 1-patch. The rules have to be applied in order. The first rule that matches determines the label of the tile. For each rule, if all the neighbours shown are present, and no previous rule matched, the tile acquires the label indicated. We can notice that the last rule consists of a single tile, so it always matches if no previous rule did. Thus every tile is assigned some label. Since the P and F metatiles are essentially the same cluster of tiles (they only differ for the matching conditions and for the shape of the boundary), these rules do not distinguish between the labels P_1 and F_1. For this reason the last rule assigns the label FP_1 to such tiles that can always be relabelled as either P_1 or F_1 later.

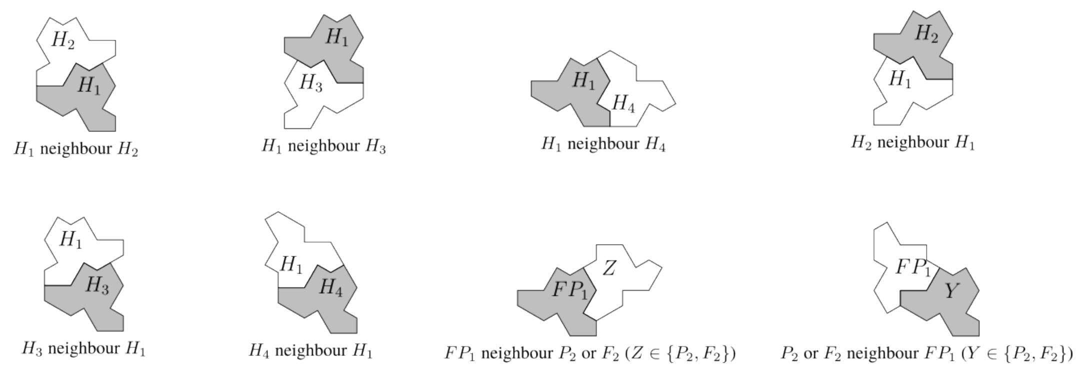


Figure 4.6: The eight within-cluster matching checks.

Now we present the within-cluster matching checks. They are shown in figure 4.6 and serve to ensure that the tessellation is divided into the four clusters. We need to apply these checks to the central tile of each patches. For example, if the central tile is labelled H_1, then it must be surrounded by H_2, H_3 and H_4 in the correct position and orientation as in the metatile H. In each diagram there are a shaded tile that represents the central tile and a labeled neighbour. Given a central tile labelled L, we have to apply all the checks in which the central tile has the same label L and verify that the neighbour tile has the correct label in the correct position and orientation. Like before, these checks do not distinguish between P_1 and F_1, it suffices to check that an FP_1 tile has either of P_2 or F_2 as its neighbour.

In figures 4.7, 4.8 are shown the between-cluster matching checks. They need to ensure that the clusters respect the matching conditions. They involve tiles of different clusters next to each other. In each diagram there are a labeled shaded tile from a cluster C and its labeled neighbour from another cluster C'. They represents a tile on one side of a cluster edge and some options for a tile on the other side of that edge. Note that in some cases, there are two alternatives for the same edge. In this case only one of those alternatives needs to pass the check.

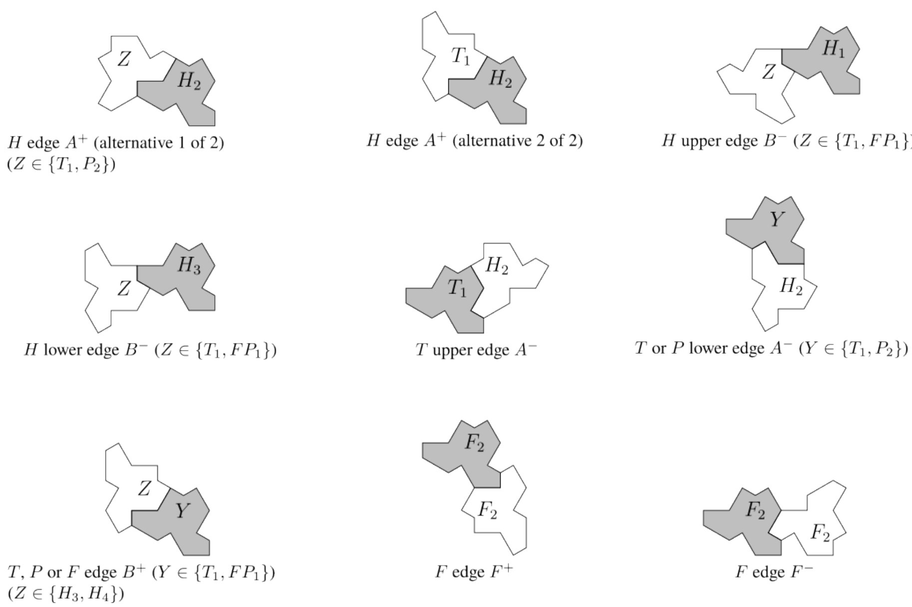


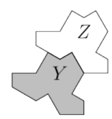
Figure 4.7: The twentyone between-cluster matching checks. (Part 1)

Since all the patches passed the within-cluster matching checks, we only need to verify that, in each patches, the central tile has a neighbour from another cluster with the correct label and in the correct position and orientation. Then it follows that the two clusters respect the matching conditions for the metatiles.

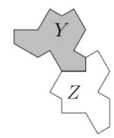
We can now continue by showing how these rules and these checks work with a surroundable 2-patch.

As said above, we need to generate a surroundable 2-patch to work with. We can extract such a patch from a larger one. Figure 4.9 shows on the left, a surroundable 2-patch selected from the partial tiling of figure 4.3. The patch has been rotated to

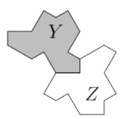
X^+ edge at top of polykite
($Y \in \{H_2, P_2, F_2\}$) (alternative 1 of 2)
($Z \in \{H_2, P_2\}$)



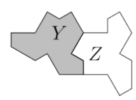
X^+ edge at top of polykite
($Y \in \{H_2, P_2, F_2\}$) (alternative 2 of 2)
($Z \in \{H_3, H_4, FP_1, F_2\}$)



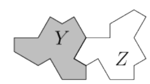
X^+ edge at right of polykite
($Y \in \{H_3, H_4, FP_1\}$) (alternative 1 of 2)
($Z \in \{H_2, P_2\}$)



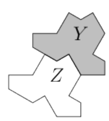
X^+ edge at right of polykite
($Y \in \{H_3, H_4, FP_1\}$) (alternative 2 of 2)
($Z \in \{H_3, H_4, FP_1, F_2\}$)



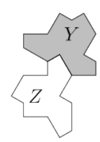
X^- edge at right of polykite
($Y \in \{H_2, P_2\}$) (alternative 1 of 2)
($Z \in \{H_2, F_2, P_2\}$)



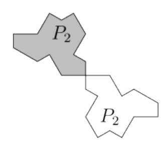
X^- edge at right of polykite
($Y \in \{H_2, P_2\}$) (alternative 2 of 2)
($Z \in \{H_3, H_4, FP_1\}$)



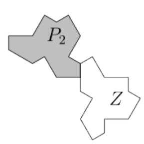
X^- edge at bottom of polykite
($Y \in \{H_3, H_4, FP_1, F_2\}$) (alternative 1 of 2) ($Z \in \{H_2, F_2, P_2\}$)



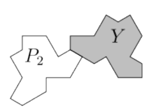
X^- edge at bottom of polykite
($Y \in \{H_3, H_4, FP_1, F_2\}$) (alternative 2 of 2) ($Z \in \{H_3, H_4, FP_1\}$)



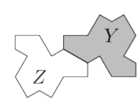
L edge at right of polykite (alternative 1 of 2)



L edge at right of polykite (alternative 2 of 2) ($Z \in \{FP_1, F_2\}$)



L edge at bottom of polykite
($Y \in \{FP_1, F_2\}$) (alternative 1 of 2)



L edge at bottom of polykite
($Y \in \{FP_1, F_2\}$) (alternative 2 of 2)
($Z \in \{FP_1, F_2\}$)

Figure 4.8: The twentyone between-cluster matching checks. (Part 2)

simplify the application of the rules and checks to the central tile. In the center, tiles belonging to different concentric rings have distinct colors and those that need to be labeled have been numbered. On the right instead, we can see the result of the application of the classification rules to the central tile and its neighbours. In order to compare a tile with the rules, we should rotate the patch, having the tile considered in the same orientation of the labeled tile in the rule, and if the tile considered is reflected, we should also reflect the patch.

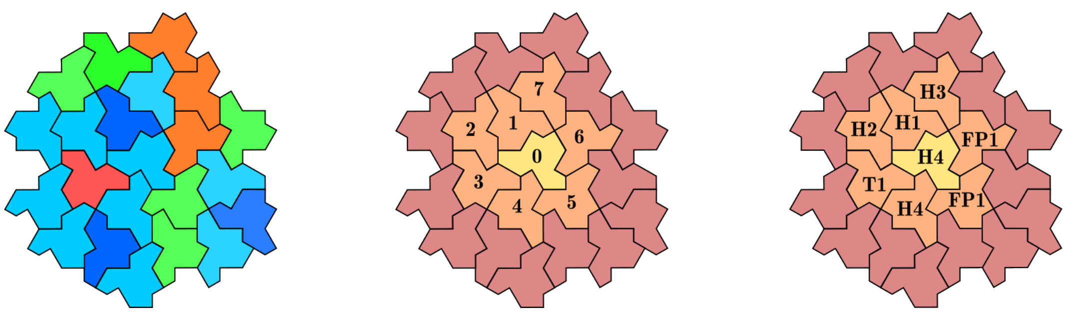


Figure 4.9

The central tile only satisfies the "#4 rule", so it has a label H_4. Likewise tile number 1 satisfies the "#1 rule" (after beeing reflected) thus it has a label H_1, and so on.
Now we have to apply the within-cluster matching checks to the central tile. Since our tile has a H_4 label it needs to satisfy only the "H_4 neighbour H_1" check, and it does. Note that, for the metatile H, these checks define a spanning tree that connects H_1 to its three neighbours.
Finally we need to apply the between-cluster matching checks to the central tile. In this case, we must first identify all the diagrams where the shaded tile has the label H_4, then for those that match our patch, we must check that the label we assigned to the neighbour tile is present in one of the options of the check. As we can see, there are four checks where the shaded tile has the label H_4 as an option, but only the "X^+ edge at right of polykite alternative 2" and the "X^- edge at bottom of polykite alternative 2" diagrams match our patch. Our central tile satisfies both of the checks, the first one with the tile number 5 labeled FP_1 and the second one with the tile number 4 labeled H_4. Thus the surroundable 2-patch considered here

successfully passed all the checks.

4.3 Supertiles

Now we want to show that in any tilings, the four metatiles T, H, P and F are forced to form combinatorial copies of themself at larger and larger scales, called *supertiles*, in the sense that these supertiles have combinatorially equivalent matching conditions of the metatiles.

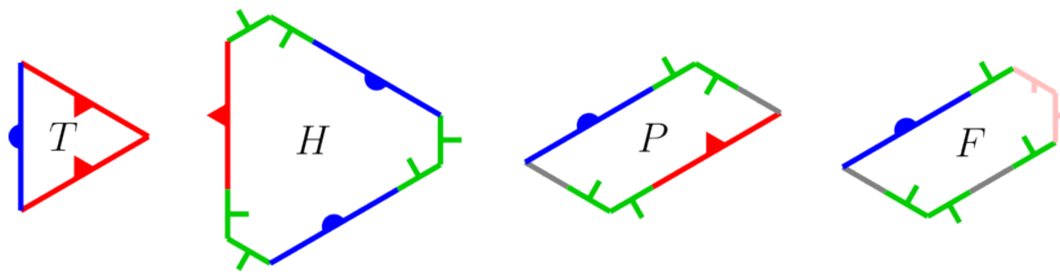


Figure 4.10: Metatiles T, H, P and F with new decorations.

It is convenient to change the representation of the metatiles as shown in the figure 4.10. In particular, we color the labeled segments A in red, B in blue, X in green, F in pink and L in gray, adding geometric decorations to distinguish the signs of the label (outward on the $+$ side, inward on the $-$ side). As before the tiles can only be rotated, not reflected.

At this point, we study how the metatiles can be put together with a cases analysis in diagrammatic form. Each diagram has unnumbered tiles that define the case being considered, and some numbered tiles which are forced by the configuration considered. Sometimes it can happen that at certain positions there are multiple choises of tiles. In this case, we mark that position with a filled black circle and create new diagrams for each choise, where the previous forced tiles are now unnumbered, but newly forced tiles are numbered.

We can start by noticing that the diagram shown in figure 4.11, called PP, cannot appear in any tilings by metatiles. In fact after adding the two forced H metatiles, nothing fits at the marked point.

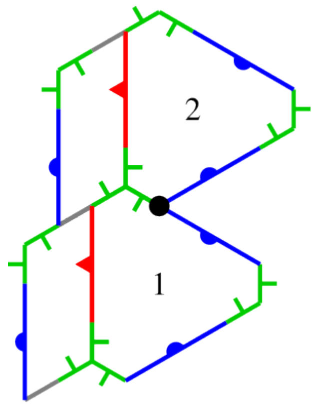
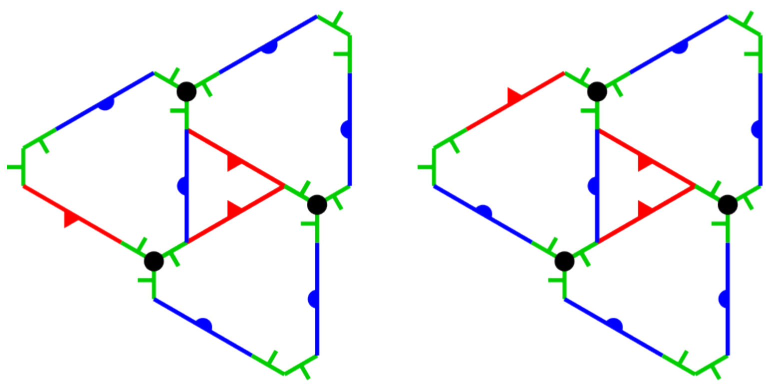


Figure 4.11: The impossible PP configuration.

The first type of diagrams we analyze are those concerning the metatile T. As we can see, the T metatile can only be adjacent to the H metatile. Since H has two blue edges and one red edge, then we have two cases for the configuration around a T tile, that we refer to as T_1 and T_2.



Case $T1$ Case $T2$

Figure 4.12: Cases T_1 and T_2.

Figure 4.12 shows these two cases. Note that they only differ on the blue edge of T with an H metatile in two different orientations. In both cases the three marked points can be filled with H or P. On a side of the figure where there are two blue edges, we must put a P, because an H metatile would result in a 60 degree angle between the two blue edges which cannot be filled. So in those sides we are forced to put a P, while in the other side we can have either a P or an H in both configurations. Thus we get four cases, that we refer to as T_1P, T_2P, T_1H and T_2H. Only the case T_1 can be extended, the other three cases are eliminated because, by putting forced tiles, the impossible PP configuration appears at the marked points.

In figure 4.13 we can see the T_1P case that has a marked point which can be filled with an F or a P resulting in cases we refer to as T_1PF and T_1PP, while in figure 4.14, 4.15 and 4.16 are shown the eliminated cases.

Only the case T_1PF (figure 4.17) can occur in any tiling by metatiles, while case T_1PP (figure 4.18) is eliminated because the PP configuration arises. So the T metatile must appear only in configuration T_1PF. This ends the cases concerning the metatile T.

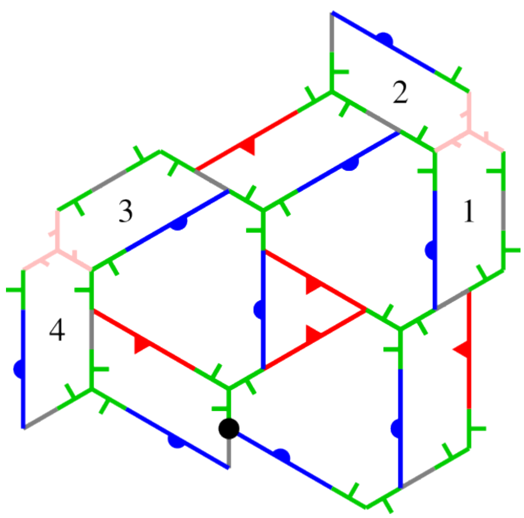


Figure 4.13: Case T_1P.

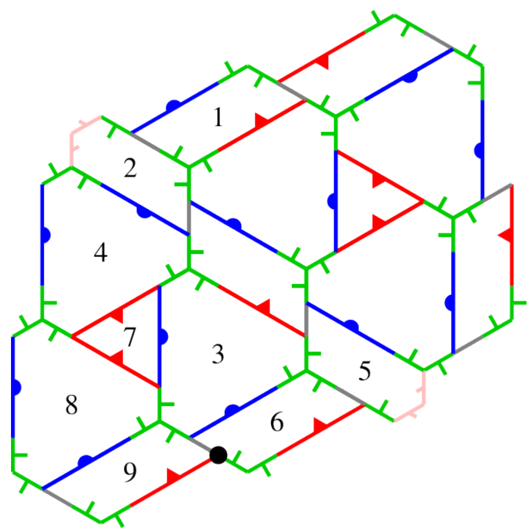


Figure 4.14: Case T_2P.

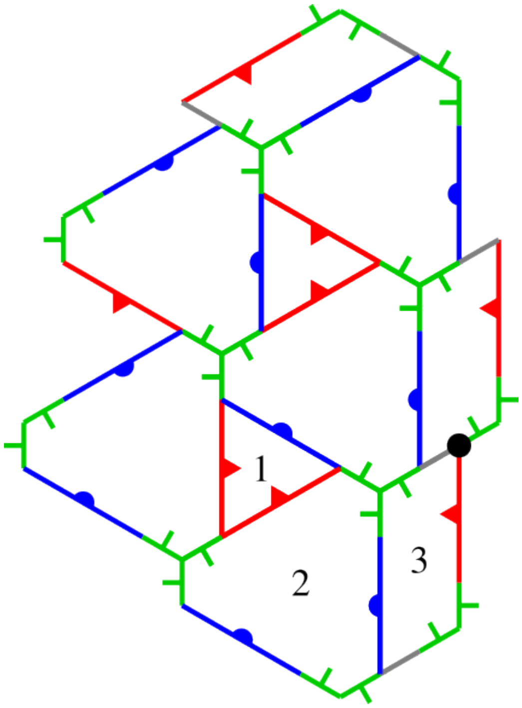


Figure 4.15: Case T_1H.

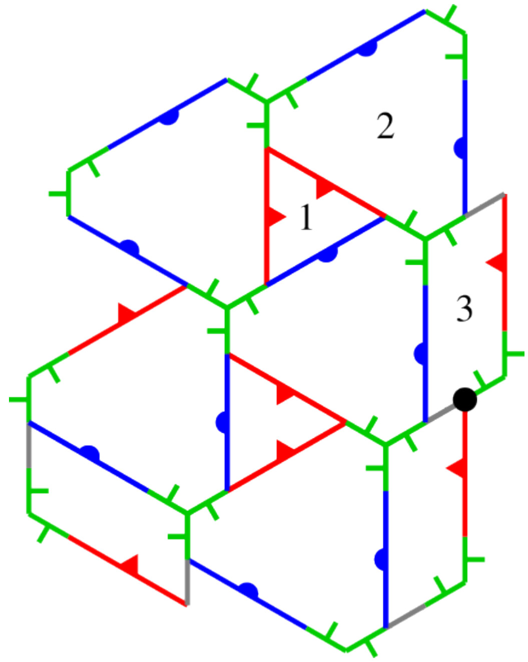


Figure 4.16: Case T_2H.

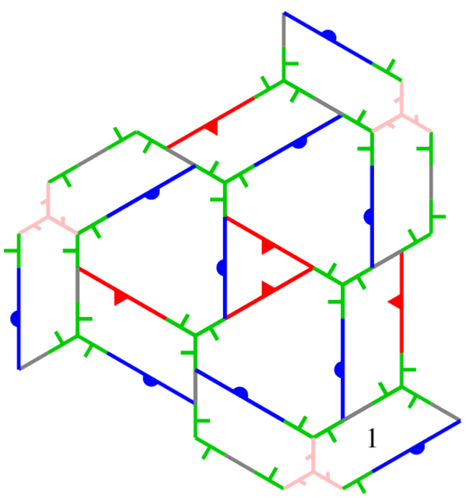


Figure 4.17: Case T_1PF.

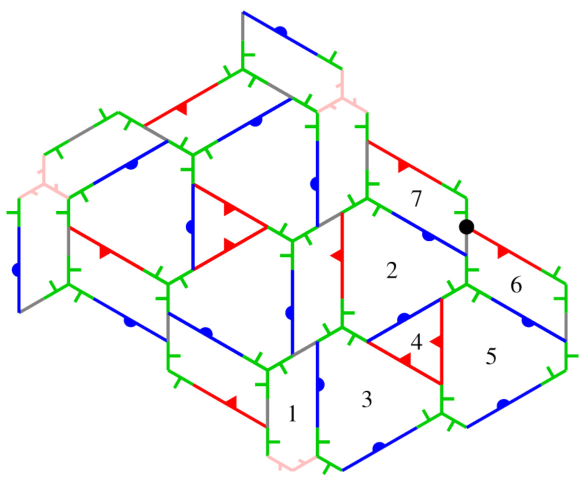


Figure 4.18: Case T_1PP.

We can now proceed to study the cases with H not adjacent to T. In this case the red edge of H can only be adjacent to a P metatile, while both blue edges can either be adjacent to a P or an F. So we have four cases, that we call HPP (figure 4.19), HPF (figure 4.20), HFP (figure 4.21) and HFF (figure 4.22). Here all the configurations are legit, the only thing to notice is that the HPF is superabundant, in the sense that it must appear in HFP or HFF configurations, depending on what we put next to the newly added H metatile (P to get HFP or F to get HFF).

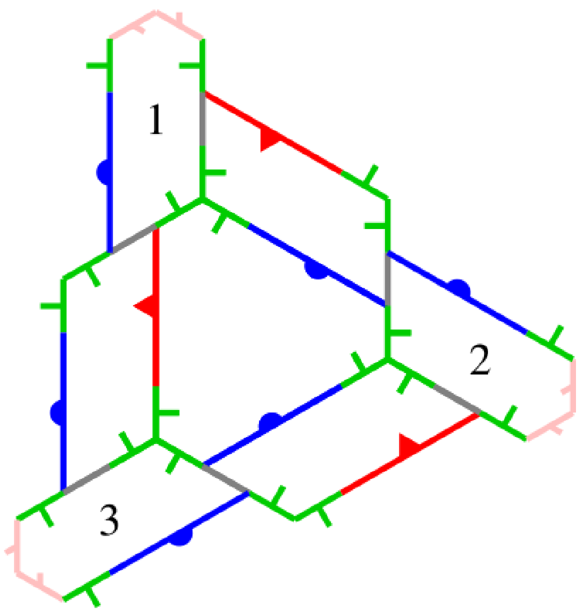


Figure 4.19: Case HPP.

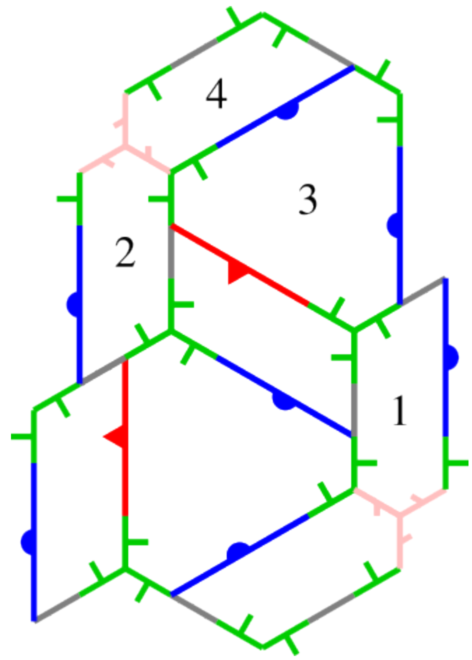


Figure 4.20: Case HPF.

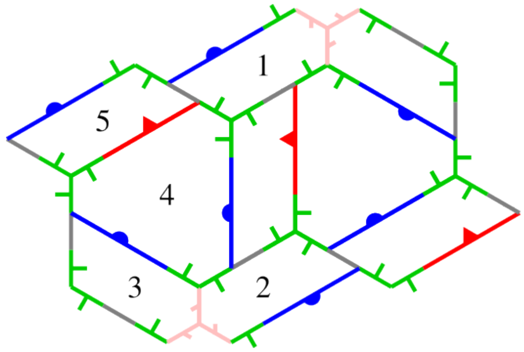


Figure 4.21: Case HFP.

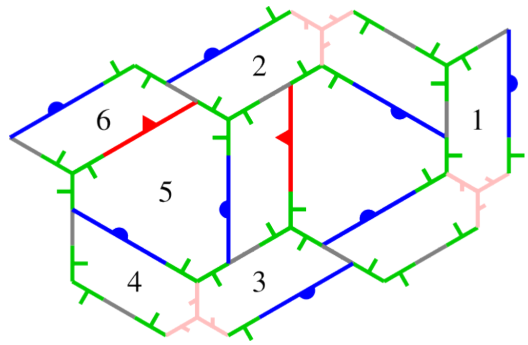


Figure 4.22: Case HFF.

There are no more diagrams to be analyzed, so the only possible cases are T_1PF, HPP, HFP and HFF. Those cases give us the so-called *supertiles* that we call H', T', P' and F' respectively, but in order to define them properly we need to do a further step. It consists on cutting in half some tiles P and F attached to them, otherwise if we try to tile with the supertiles, some P and F will overlap perfectly occupying the same region of the plane. The bisection of tiles P and F is shown in figure 4.23. We divide the P into P^+ (the sub-tile that has a blue edge) and P^-, and the F into G^+ (the sub-tile that has a blue edge) and G^-.

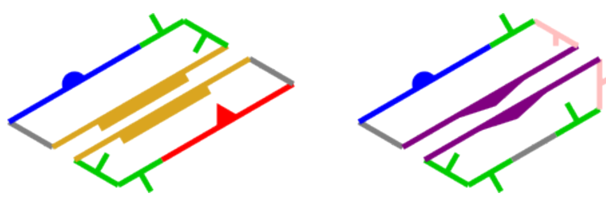


Figure 4.23: The bisection of metatiles P and F.

We can now present the supertiles H', T', P' and F' that are shown in figure 4.24-4.27 , where we labeled the edges of the bisected P and F tiles. These new labels give the matching conditions for the supertiles, where A_2^+ must be adjacent to A_2^-, B_2^+ to B_2^-, X_2^+ to X_2^-, F_2^+ to F_2^-, and L_2 to L_2.
We can also give precise rules to allocate the metatiles into the supertiles.

- Each T tile is allocated to an H' supertile.

- Each H tile in case HPP is allocated to a T' supertile.
- Each H tile in case HFP is allocated to a P' supertile.
- Each H tile in case HFF is allocated to an F' supertile.
- Each H tile in case HPF was allocated to a supertile by exactly one of the previous two rules.
- Each half of a P tile, and each G^+ tile, is adjacent to exactly one H tile along its A^- or B^+ edge, and is allocated to the same supertile as that H tile.
- Each G^- tile is allocated to the same supertile as the H tile adjacent to the X^-.

Note that for the last rule to be well defined, we need to show that this X^- edge of G^-, the one between an L and an F^+, is indeed adjacent to a H tile. The only other possibility not violating the metatile matching conditions is shown in figure 4.28. This configuration is not possible, because no tile can be adjoined at the marked point.
At this point in the original article is shown that the supertiles must adjoin each other in accordance with the matching conditions and that they are fully combinatorially equivalent to the metatiles so the composition of metatiles into supertiles may be applied n times for all n. So we have established the the following result: [6]

Proposition 3 *In any tiling by the four metatiles, after bisecting P and F metatiles as described above, the metatiles fit together to form larger, combinatorially equivalent supertiles, thereby forming a substitution system. The tiling by the supertiles has the same symmetries as the tiling by the metatiles.*

We can now use the results obtained to prove the following theorem:

Theorem 6 *The hat polikite is an aperiodic monotile.*

Proof.
Let $\mathcal{T}$ be a tiling by hat polykites. From proposition 2 it follows that $\mathcal{T}$ can be uniquely divided into the four clusters of figure 4.4 satisfying the given matching conditions, with the resulting tiling by metatiles having the same symmetries as $\mathcal{T}$. Then from proposition 3 it follows that these four metatiles must follow a substitution system forming combinatorially equivalent supertiles. So each metatiles belong uniquely to a level-1 supertile with the same combinatorial structure as the metatiles. Then the level-1 supertiles must therefore belong uniquely within level-2 supertiles inheriting the same combinatorial structure, and so on for subsequent levels. Thus any tiling by metatiles must be non-periodic, because (using an argument similar to lemma 1) if it had a translational symmetry, then for sufficently large k there would exist a level-k supertile that overlaps its image under this translation. So any metatile in the intersection of these two supertiles would then lie within both of their infinite hierarchies, but this is absurd thanks to the uniqueness of these hierarchies. Since $\mathcal{T}$ has the same symmetries as the tiling by metatiles, then $\mathcal{T}$ is non-periodic. Moreover since it's possible to construct partial tilings by metatiles of arbitrary size, then from the Extension Theorem [2] it follows that the metatiles and hence the hats can tile the plane. Thus the hat polykite is an aperiodic monotile. □

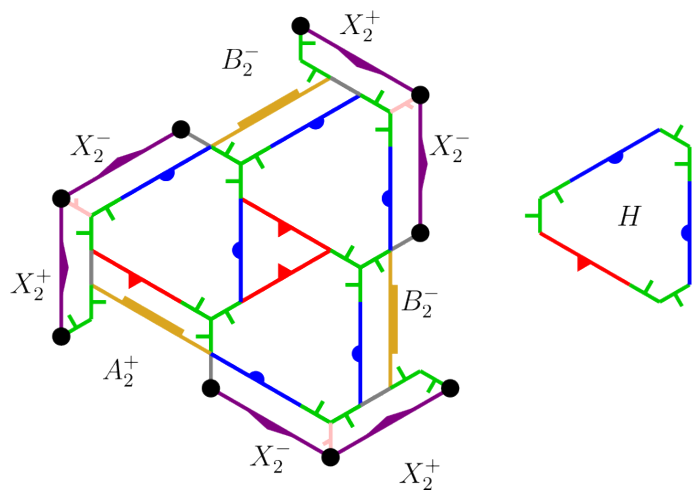


Figure 4.24: Supertile H', alongside corresponding H.

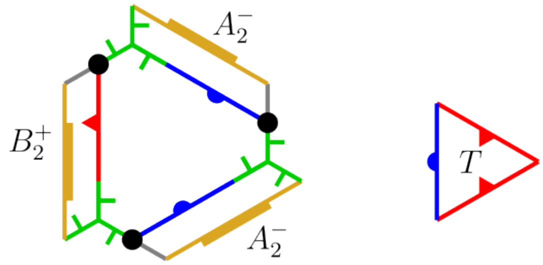


Figure 4.25: Supertile T', alongside corresponding T.

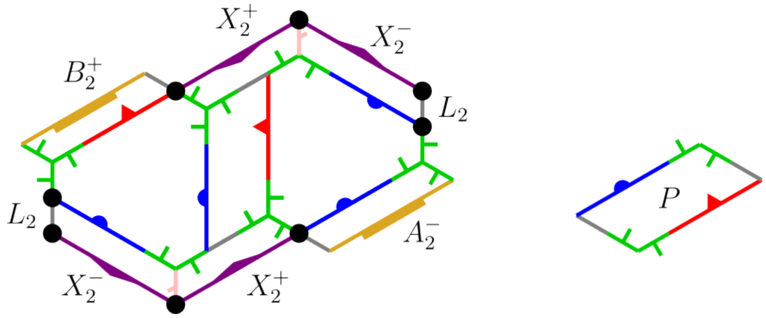


Figure 4.26: Supertile P', alongside corresponding P.

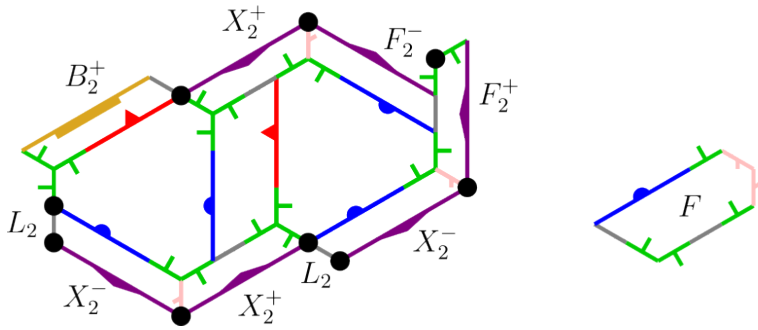


Figure 4.27: Supertile F', alongside corresponding F.

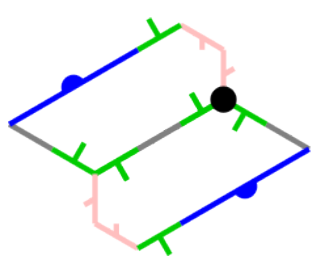


Figure 4.28: Impossible configuration of two F tiles.

Appendix A

Hilbert's axioms

Undefined Terms: point, line, plane, lie (or lie on), between, congruent

I. Axioms of Incidence (Connection)

I.1 Through any two distinct points A and B there exists a line l.

I.2 Through any two distinct points A and B there exists no more than one line l.

I.3 On every line there exist at least two distinct points. There exist at least three points that do not lie on the same line.

II. Axioms of Order (Betweenness)

II.1 If point B lies between points A and C, then A, B and C are distinct points of a line, and B lies between C and A.

II.2 For two distinct points A and C, there is at least one point B on the line AC such that C lies between A and B.

II.3 For three distinct points A, B and C on a line, there is one and only one point which lies between the other two.

II.4 (Pasch's Axiom) Let A, B and C be three distinct points that do not lie on a line, and let l be a line in the plane that does not meet any of the points A, B

or C. If line l passes through a point of segment AB, it also passes through a point of segment AC or a point of segment BC.

III. Axioms of Parallels

III.1 Let l be a line and A a point not on line l. Then there is at most one line in the plane, determined by l and A, that passes through A and does not intersect l.

IV. Axioms of Continuity

IV.1 (Archimedes' Axiom) If AB and CD are any segments, then there exists a number n such that n segments CD constructed contiguously along ray $\vec{AB}$ starting at A, will pass beyond B.

IV.2 (Axiom of Line Completeness) An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I, II, V and from IV.1 is impossible.

V. Axioms of Congruence

V.1 If A and B are two distinct points on line l, and if A' is a point on the same or on another line l', then it is always possible to find a point B' on a given side of the line l' such that $AB \cong A'B'$.

V.2 For segments AB, $A'B'$ and $A''B''$, if $A'B' \cong AB$ and $A''B'' \cong AB$, then $A'B' \cong A''B''$, or briefly, if two segments are congruent to a third one they are congruent to each other.

V.3 On a line l, let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or another line l', let $A'B'$ and $B'C'$ be two segments which except for B' also have no point in common. If $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.

V.4 Given rays $\vec{AB}$ and $\vec{AC}$ which lie on distinct lines, and ray $\vec{A'B'}$, there is exactly one ray $\vec{A'C'}$ on each side of line $A'B'$ such that $\angle B'A'C' \cong \angle BAC$. Every angle is congruent to itself.

V.5 If for two triangles $\triangle ABC$ and $\triangle A'B'C'$ we have $AB \cong A'B'$, $AC \cong A'C'$, $\angle BAC \cong \angle B'A'C'$, then we also have $\angle ABC \cong \angle A'B'C'$.

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