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Configuration spaces and braids on graphs

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Introduction

The date of birth of braids theory is commonly set in 1925, when Emil Artin in his work [Ar25] provided a geometrical definition and a presentation for the n-braid group $\mathcal{B}_n$, but the notion of braid was already well known.
The first sketch of a braid can be found in Gauss' notebooks dating back to the first half of the XIX century. Moreover, Gauss was the first to pose the problem of classifying braids up to a suitable equivalence relation.
Later, some other authors including Hurwitz, introduced implicitely in their works the concept of braid groups without providing a formal definition of the structure until Artin did it.
The emphasis of Artin work lies in translating geometrical issues into group-theoretical ones, indeed for example the classification of braids up to isotopy is seen as the word problem for the braid group.

In [Ar], Artin showed also the relation between the braid group $\mathcal{B}_n$ and the configuration space of n points on the plane $\mathbb{R}^2$. Indeed, consider the loops based at a point ω in the configuration space $\mathcal{C}_n(\mathbb{R}^2)$, consisting of n points whose second coordinate is zero, $\omega = \{(1,0),(2,0),\dots,(n,0)\}$. Then, suppose that at some moment t, a loop passes through an element $x(t) \in \mathcal{C}_n(\mathbb{R}^2)$ and notice that $x(t)$ is a plane with n distinct points marked on it. If we place the plane $x(t)$ in $\mathbb{R}^3$ by adding the third coordinate $z = t$ and if we let t vary in $[0,1]$, then we obtain an n-braid. Hence, there is a bijective correspondence between the homotopy classes of loops based at ω and the isotopy classes of n-braids, that is $\mathcal{B}_n = \pi_1(\mathcal{C}_n(\mathbb{R}^2))$.

Many applications of braids theory have been found out during the last century, but we are going to focus on an unexpected one in robotics.
In the $1990'$s, some mathematicians approached to safe control schemes for automated guided vehicles (AGVs). The problem was designin a control scheme which avoids collissions with obstacles or other AGVs and, at the same time, guarantees a high enough efficiency in completing the assigned task.
The workspace floor of the factory with n AGVs moving, can be thought

of as the configuration space $\mathcal{C}_n(\mathbb{R}^2)$, but in order to reduce the sophistication required for the AGVs, they can be imagined to move only on guidepath wires. For this reason, the problem was moved to configuration spaces on graphs and consequently to the braid groups on graphs.

In the last two decades, different methods have been proposed in order to provide increasingly easier and efficient computing of presentations for braid groups on graphs.

The pioneering works by Ghrist and Abrams [Gh99], [Ab], [AG], [Gh07] have become the basis of further results. The so called Subdivision Theorem, proved by Abrams in [Ab], was used by Farley and Sabalka in [FS05], [FS09] together with the discrete Morse theory to give a description of the critical cells of a "discretization" of the configuration space $\mathcal{C}_n(G)$ of a graph G admitting a cubical complex structure. In particular, this discretized configuration space requires a subdivision of the graph G depending on the number n of points on it. Then, Farley and Sabalka were able to compute a presentation for the braid group $\mathcal{B}_n(G)$ of G where the generators are the critical 1-cells and the relations are given by the critical 2-cells.

The purpose of this thesis is to study the configuration spaces of graphs and to compute presentations for the corresponding braid groups, adopting a simpler approach that avoids the need for the subdivision theorem and the Morse theory. Our starting idea was to explicitely construct a kind of normalized configuration subspace $\mathcal{N}_n(G)$ which is a weak deformation of $\mathcal{C}_n(G)$, and a homeomorphism between $\mathcal{N}_n(G)$ and a cubical complex $\mathcal{Q}_n(G)$ without requiring any subdivision of G. Then, a presentation for $\mathcal{B}_n(G)$ can be directly derived from the 2-skeleton of $\mathcal{Q}_n(G)$, without using Morse theory.

At a later stage we found out that in [Sw], Swiatkowski had already followed a similar argument which was not cited in the later works by Ghrist and Abrams [AG], [Gh07]. Indeed, he defined an embedding i of a cubical complex $K_n(G)$ into the configuration space $\mathcal{C}_n(G)$ and then he stated that there is a certain strong deformation retraction $r\colon \mathcal{C}_n(G) \to i(K_n(G))$. All the proofs are left to the reader and we have verified that r is not even a retraction.

Chapter 1 reviews the basic notions and results regarding graphs, CW-complexs, fundamental groups and group presentations, which will be needed for later chapters. The last two sections briefly introduce classical and discrete Morse theory, stating the main theorems of both theories.

In chapter 2, first we define configuration spaces and braid groups on metric spaces, then the last section is focused on describing in some detail the results achieved by Ghrist, Abrams, Farley and Sabalka regarding the presentations for braid groups on graphs.

Chapter 3 is subdivided in three sections as follows. First we define the normalized configuration space $\mathcal{N}_n(G)$, consisting of all the configurations $x \in \mathcal{C}_n(G)$ whose first and last points along each edge e of G vary inside two proper intervals, while the intermediate points are uniformly distributed between them.
Then, we construct a continuous mapping $\Phi\colon \mathcal{C}_n(G) \to \mathcal{C}_n(G)$ such that $\operatorname{Im}\Phi = \mathcal{N}_n(G)$ and we prove that Φ gives a weak deformation of the configuration space $\mathcal{C}_n(G)$ into the normalized configuration space $\mathcal{N}_n(G)$.
Finally, we construct a cubical complex $\mathcal{Q}_n(G)$ homeomorphic to $\mathcal{N}_n(G)$, and provide a presentation for the braid group $\mathcal{B}_n(G)$ as the fundamental group of $\mathcal{Q}_n(G)$.

In chapter 4, we analyze in detail some families of graphs and we compute a presentation for their braid groups, based on the results seen in chapter 3. In particular, we find general formulas for the radial trees and the bouquets of loops, which agree with the already known results seen in chapter 2.

Chapter 1
Preliminaries

In this preliminary chapter we introduce notions and tools necessary to understand later results. In particular, we need to deal with cell complexes, cubical and simplicial complexes, graphs, discrete Morse theory and some other topics from algebraic topology.

Let $\mathbb{R}^n = \{(x_1, \ldots, x_n) \ : \ x_i \in \mathbb{R} \ \forall \ i = 1, \ldots, n\}$ be the n-dimensional Euclidean space provided with the usual metric $d(x, y) = \|x - y\|$ and the topology induced by this metric.
We adopt the following notations:
$B^n = \{x \in \mathbb{R}^n \ : \ \|x\| \leq 1\}$ for the closed unit n-ball in $\mathbb{R}^n$;
$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ for the unit n-sphere in $\mathbb{R}^{n+1}$;
$\operatorname{Int} B^n = B^n \setminus S^{n-1}$ for the interior of $\mathcal{B}^n$;
$\operatorname{Bd} B^n = S^{n-1}$ for the boundary of $\mathcal{B}^n$.

1.1 Graphs

Definition 1.1.1. A finite graph $G = (V_G, E_G, F_G)$ consists of a set V_G of *vertices*, a set E_G of *edges* and a map $F_G \colon E_G \to (V_G \times V_G)/\Sigma_2$ which associates to each edge $e \in E_G$ an unordered pair of non-necessarily distinct vertices in V_G, called the *endpoints* of e.
When the endpoints of an edge $e \in E_G$ coincide, the edge e is said a *loop.*

The *degree* or *valence* of a vertex v, denoted by $\deg(v)$, is the number of occurencies of v in the pairs of endpoints of all the edges. A vertex of degree 0 is an *isolated* vertex, a vertex of degree 1 is a *terminal* vertex and a vertex of degree $= 2$ is an *inessential* vertex.

A *planar graph* is a graph which can be embedded into $\mathbb{R}^2$.

A *subgraph* of a graph $G = (V_G, E_G, F_G)$ is a graph $H = (V_H, E_H, F_H)$ such that $V_H \subset V_G$, $E_H \subset E_G$ and $F_H = F_{G|E_H}$.

A *path* in a graph G is a sequence of edges $P = e_1, e_2, \dots, e_m$ for which there exists a sequence of vertices $v_0, \dots, v_m$ of G so that $F_G(e_j) = [v_{j-1}, v_j]$. A graph G is *connected* if for each pair of vertices $v, w \in V_G$ there exists a path as above such that $v_0 = v$ and $v_m = w$.
A path P is *closed* if $v_m = v_0$. A *cycle* is a closed path such that no vertex appears more than once except for $v_m = v_0$.

A *tree* is a simply connected graph [Sp], or equivalently a graph without any cycle. A tree is *linear* if every vertex of degree strictly greater than 2 lies along a single embedded arc.

A *spanning tree* of a graph G is a subgraph H of G which is a tree and such that $V_H = V_G$.

Proposition 1.1.2. *Every connected graph G has a spanning tree.*

Proof. We argue by induction on the number $n \geq 0$ of edges of G. If $n = 0$, then G consists of one vertex and so it is already a spanning tree of itself. If $n > 0$, then either G is a tree, and so it is a spanning tree of itself, or G contains a cycle. In the latter case, we can eliminate one edge of the cycle and the resulting subgraph is still connected and it has $n - 1$ edges. Hence, it has a spanning tree which is also a spanning tree for G.

□

A *topological graph* G is a topological space which comes from a graph $G = (V_G, E_G, F_G)$ by replacing each vertex v_i with a point x_i and each edge e with a copy I_e of the unitary interval $[0, 1]$ such that the endpoints of I_e are identified with x_i and x_j if $F_G(e) = [v_i, v_j]$. The topology of G is the quotient topology of the disjoint union $\{x_i\}_i \bigsqcup_{F_G(e)} I_e$.

Remark 1.1.3. Let G be a connected finite topological graph not homeomorphic to S^1 and G' be the topological graph obtained from G by eliminating all the inessential vertices and fusing every pair of edges sharing the same inessential vertex into a single edge. Then, $G \cong G'$.
Observe that if $G \cong S^1$, then all the vertices of G are inessential so if we remove all of them we do not have even a graph anymore.

Given a connected topological graph G, it is possible to think of it as a linear graph in $\mathbb{R}^n$ with $n = |V_G|$ as follows.
Assume $V_G = \{v_1, \dots, v_n\}$ and consider the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$. We put each vertex v_i of G on the head of a vector $\frac{e_i}{\sqrt{2}}$ for $i = 1, \dots, n$

and we let the edges of G be segments joining two vertices v_i, v_j. In this way each edge $e \in E_G$ has unitary length. Then, we define the metric d_G on G by setting for every $p, q \in G$

$$d_G(p,q) := \min_{A \in \mathcal{A}_{pq}} L(A),$$

where $\mathcal{A}_{pq}$ is the collection of all poligonal arcs in G between p and q and L denotes the Euclidean length.
Notice that d_G is independent on the specific indexing of V_G.

1.2 Complexes

A closed *n-cell* $(\bar{c}, h_{\bar{c}})$ consists of a topological space $\bar{c}$ and a homeomorphism $h_{\bar{c}} \colon B^n \to \bar{c}$. We indicate by $\operatorname{Int} \bar{c}$ the interior of $\bar{c}$, that is the image $h_{\bar{c}}(\operatorname{Int} B^n)$ of the interior of B^n and by $\operatorname{Bd} \bar{c}$ the boundary of $\bar{c}$, that is the image $h_{\bar{c}}(\operatorname{Bd} B^n)$ of the boundary of B^n.
An open n-cell (c, h_c) consists of a topological space c and a homeomorphism $h_c \colon \operatorname{Int} B^n \to c$. Notice that the interior of a closed n-cell is an open n-cell such that $h_c = h_{\bar{c}|\operatorname{Int} B^n}$.

A finite *CW-complex* K is a topological space together with a partition of it into disjoint open cells such that:

i) K is Hausdorff;

ii) for each open n-cell $(c, h_c) \subset K$, there exists a characteristic map

$$e_c \colon B^n \to K$$

such that $e_{c|\operatorname{Int} B^n} = h_c$ and $e_c(\operatorname{Bd} B^n)$ is contained into a finite union of open cells of dimension less than n;

iii) a set A is closed in K if and only if $A \cap \bar{c}$ is closed in $\bar{c}$ for any open n-cell (c, h_c).

The finiteness condition in *ii*) is called *"closure finiteness"* and condition *iii*) determines the so called *"weak topology"* with respect to the collection of the closed cells $(\bar{c}, h_{\bar{c}})$. These two expressions are at the origin of the term "CW-complex".

The *dimension* $\dim K$ of a CW-complex K is the maximum of the dimensions of its cells.

A subspace L is a *subcomplex* of K if it is a union of cells of K which is still a CW-complex.
In particular, for every $i \leq \dim K$, the subspace K^i is constituted by the union of all the cells of K of dimension at most i and it is a subcomplex of K called the *i-skeleton* of K.

Given a topological space X and a continuous function $f\colon \operatorname{Bd} B^n \to X$, the notion of *attaching an n-cell* to X consists of the topological union $X \sqcup B^n$ quotiented out by the minimal equivalence relation which identifies each point $x \in \operatorname{Bd} B^n$ with $f(x) \in X$. The resulting space is denoted by $X \cup_f B^n$.
A finite CW-complex K can be realized by inductively constructing its skeleta. Namely, the n-skeleton K^n can be obtained by attaching each n-cell (c, h_c) of K to K^{n-1} via the attaching map $f_c = e_{c|\operatorname{Bd} B^n}\colon \operatorname{Bd} B^n \to K^{n-1}$.

Let K be a finite CW-complex of dimension n, we define the *Euler characteristic* of K, denoted by $\chi(K)$, as

$$\chi(K) = \sum_{i=0}^{n} (-1)^i n_i$$

where n_i indicates the number of i-cells of K.

Consider a unit interval I in $\mathbb{R}$ and the standard cube I^n in $\mathbb{R}^n$.
An *n-cube* (c, h_c) consists of a topological space c equipped with a homeomorphism $h_c : I^n \to c$.
An $(n-1)$-face r of I^n can be identified by two parameters k and i, where $k = 1, \ldots, n$ indicates the direction and $i = 0, 1$, such that

$$r_{k,0} = \{t \in I^n | t_k = 0\}$$

and

$$r_{k,1} = \{t \in I^n | t_k = 1\}.$$

We define a natural parametrization of the face $r_{k,i}$ as follows:

$$p_{k,i}\colon I^{n-1} \to r_{k,i} \;\text{ s.t. }\; (t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{k-1}, i, t_{k+1}, \ldots, t_{n-1}).$$

An $(n-1)$-face (d, h_d) of (c, h_c) consists of the topological space $d = h_c(r_{k,i})$ equipped with a homeomorphism $h_d : I^{n-1} \to d$ such that $h_d = h_c \circ p_{k,i}$.

Definition 1.2.1. Let (c, h_c) be an n-cell in K and (d, h_d) an $(n-1)$-face of (c, h_c) whose characteristic maps are $e_c \colon I^n \to K$ and $e_d \colon I^{n-1} \to K$ respectively.
A finite *cubical complex* $\mathcal{Q}$ is a finite CW-complex such that each n-cell (c, h_c) of $\mathcal{Q}$ is an n-cube and each $(n-1)$-face (d, h_d) of (c, h_c) is an $(n-1)$-cell of $\mathcal{Q}$ with characteristic map $e_d = e_{c|r_{k,i}}$, up to Euclidean isometries of I^{n-1}.

The *standard n-simplex* Δ^n is a subset of $\mathbb{R}^{n+1}$ such that

$$\Delta^n = \Big\{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \ \ \forall i = 0, \dots, n\Big\}.$$

Consider an $(n-1)$-face r of Δ^n, then it can be identified by a single parameter k indicating the direction, $k = 0, \dots, n$ such that $r_k = \{t \in \Delta^{n-1} : t_k = 0\}$. An *n-simplex* (σ, h_σ) is a topological space σ equipped with a homeomorphism $h_\sigma \colon \Delta^n \to \sigma$.
An $(n-1)$-face (ρ, h_ρ) of (σ, h_σ) is the topological space $\rho = h_\sigma(r_k)$ equipped with a homeomorphism $h_\rho \colon \Delta^{n-1} \to \rho$. It can be given a definition of simplicial complexes analogous to that of cubical complex.

Definition 1.2.2. A finite *simplicial complex* K is a CW-complex such that each n-cell (σ, h_σ) of K is an n-simplex and each $(n-1)$-face (ρ, h_ρ) of (σ, h_σ) is an $(n-1)$-cell of K with characteristic map $e_\rho = e_{\sigma|r_k}$, up to isometries of Δ^{n-1}.

Now consider $\mathbb{R}^n$, and let $\{v_1, \dots, v_m\}$ be a set of m affinely independent points with $m \leq n$. An m-simplex σ in $\mathbb{R}^n$ spanned by $v_1, \dots, v_m$ is a subset of $\mathbb{R}^n$ such that

$$\sigma = \Big\{x \in \mathbb{R}^m : \ x = \sum_{i=1}^{m} t_i v_i \quad \text{where} \quad \sum_{i=1}^{m} t_i = 1, t_i \geq 0 \text{ for all } i\Big\}.$$

A 0-simplex in $\mathbb{R}^n$ is just a point v_1. A 1-simplex spanned by v_1, v_2 is the line segment joining v_1 and v_2. A 2-simplex spanned by v_1, v_2, v_3 is the triangle with vertices v_1, v_2, v_3 and so on.

Any non-empty subset of $\{v_1, v_2, \dots, v_m\}$ of cardinality $p \leq m$ spans a p-simplex ρ called a *p-face* of σ. We denote this by $\rho \leq \sigma$.
A simplex τ is a *coface* of a simplex σ if σ is a face of τ.

Definition 1.2.3. A finite *simplicial complex* K in $\mathbb{R}^n$ is a subspace given by the union of a finite collection of simplices in $\mathbb{R}^n$ such that

i) if $\rho \leq \sigma$ and $\sigma \in K$ then also $\rho \in K$, i.e. every face ρ of a simplex σ in K is a simplex in K itself;

ii) if $\sigma, \tau \in K$ then $\sigma \cap \tau$ is empty or a face of both σ and τ, and so it is itself a simplex in K.

Remark 1.2.4. Observe that according to Def. 1.2.3, a simplical complex in $\mathbb{R}^n$ is univocally determined by its vertices, while in Def. 1.2.2 there was the possibility to have distinct simplices sharing the same vertices, indeed the characteristic maps were not necessarily injective on the boundary.
This means that the two definitions are not equivalent, but a simplicial complex in $\mathbb{R}^n$ is a particular case of a simplicial complex seen as a CW-complex.

A definition equivalent to Def.1.2.3 is that of abstract simplicial complex.
An *abstract finite simplicial complex* $\mathcal{S}$ is a collection of finite non-empty sets such that if A is in $\mathcal{S}$, then also every non-empty subset of A is in $\mathcal{S}$.
An element A in $\mathcal{S}$ is said an *abstract simplex* of $\mathcal{S}$. The dimension of A in $\mathcal{S}$ is the cardinality of A minus 1.
The 0-simplices in $\mathcal{S}$ are the *vertices* of $\mathcal{S}$. Each simplex of $\mathcal{S}$ that is a subset of $A \in \mathcal{S}$ is called a *face* of A.

Let K be a finite simplicial complex in $\mathbb{R}^n$ and $\mathcal{S}$ be the collection of finite sets of vertices $\{v_1, \dots, v_k\}$ which are vertices of some simplex of K. Then, $\mathcal{S}$ is an abstract simplicial complex and it is called the *vertex scheme* of K. Equivalently K is said the *geometric realization* of $\mathcal{S}$. [Le]

1.3 Homotopy and fundamental group

Let X, Y be two topological spaces. A *homotopy* of X into Y is a continuous map

$$H \colon X \times [0,1] \to Y.$$

Equivalently, H can be seen as a continuous family of continuous functions $(h_t \colon X \to Y)_{t \in [0,1]}$ such that $h_t(x) = H(x,t)$ for each $x \in X$ and for each $t \in [0,1]$.

Given two continuous functions $f, f' \colon X \to Y$, then H is a *homotopy* between f and f' if $h_0 = f$ and $h_1 = f'$.
If there exists a homotopy between f and f', then the functions f and f' are said *homotopic* and we write $f \simeq f'$.
It can be easily verified that homotopy relation $\simeq$ is a compositive equivalence relation.

Two topological spaces X and Y are said *homotopy equivalent* if there exist two continuous functions $f \colon X \to Y$ and $g \colon Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

If X is homotopy equivalent to a single point, then X is said *contractible.*

Definition 1.3.1. A *weak deformation* of a topological space X into a subspace A of X is a homotopy

$$H \colon X \times [0,1] \to X$$

such that the following are satisfied:

i) $H(x, 0) = x$ for all $x \in X$,

ii) $H(x, 1) \in A$ for all $x \in X$,

iii) $H(x, t) \in A$ for all $x \in A$ and for all $t \in [0, 1]$.

Then A itself is said a *weak deformation* of X.

A *strong deformation* of X onto A is a weak deformation which satisfies also

$$H(x, t) = x \text{ for all } x \in A \text{ and for all } t \in [0, 1].$$

Then A itself is said a *strong deformation* of X.

Equivalently, we can say that A is a weak deformation of X if there exists a continuous map $r \colon X \to A$ such that $r \circ i \simeq id_A$ and $i \circ r \simeq id_X$ where $i \colon A \to X$ is the inclusion map. Hence, A and X are homotopy equivalent.

Let X be a topological space, $*$ a point in X and $\Omega(X, *)$ the set of loops in X based at point $*$:

$$\Omega(X, *) = \{\omega \colon [0,1] \to X \text{ s.t. } \omega \text{ continuous and } \omega(0) = \omega(1) = *\} .$$

Then, the set $\Omega(X, *)$ equipped with the concatenation of loops and quotiented out by homotopy relation $\bmod \{0, 1\}$ forms a group called the

fundamental group of $(X, *)$, which is an homotopy invariant. We denote it by
$$\pi_1(X, *) = (\Omega(X, *), \cdot)/ \simeq_{\{0,1\}} .$$

Notice that if X is a path connected topological space and $*$, $*'$ are two points in X, then there is a path between $*$ and $*'$ and it can be induced an isomorphism between $\pi_1(X, *)$ and $\pi_1(X, *')$. Thus, the fundamental group $\pi_1(X, *)$ is independent on the choice of the base point $*$. For this reason, from now on we are going to use the notation $\pi_1(X)$ instead of $\pi_1(X, *)$ while considering path connected topological spaces.

The first *homology group* $H_1(X)$ is the abelianization of the fundamental group $\pi_1(X)$ and the Euler characteristic of X is related to the homology groups as follows
$$\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X).$$

Let us remind an essential result for computing fundamental groups of path connected topological spaces.

Theorem 1.3.2. (Seifert-Van Kampen Theorem) [Mu] *Let X be the union of two open, path connected subsets X_1 and X_2 whose intersection $X_1 \cap X_2$ is non-empty and path connected. Let x_0 be a point in $X_1 \cap X_2$ and let*
$$j_1 \colon X_1 \cap X_2 \to X_1 \quad \text{and} \quad j_2 \colon X_1 \cap X_2 \to X_2$$
be the embeddings into X_1 and X_2 respectively.
Then,
$$\pi_1(X) \cong \frac{\pi_1(X_1) * \pi_1(X_2)}{N(\{j_1(\omega) j_2(\omega)^{-1},\ \omega \in \pi_1(X_1 \cap X_2)\})}.$$

Proposition 1.3.3. *A CW-complex X is connected if and only if its 1-skeleton X^1 is connected.*

Proof. First observe that attaching an m-cell (c, h_c) with $m > 1$ to any space X does not change the number of components of the space. Indeed, considering the characteristic map $e_c \colon B^m \to X$, we can notice that $e_{c|\operatorname{Bd} B^m}$ is continuous and its image must be entirely contained in a single connected component of X since $\operatorname{Bd} B^m$ is connected.
Then, the proposition immediately follows by induction on the number $n \geq 0$ of the cells of X whose dimension is greater than 1. □

Lemma 1.3.4. *Let X be a path connected space and $X' = X \cup_{f_c} c$, with (c, h_c) an i-cell with $i \geq 3$. Then, $\pi_1(X) \cong \pi_1(X')$.*

Proof. First notice that X' is connected by the previous proposition.
Let us consider the characteristic map $e_c \colon B^i \to c$ and a loop

$$\omega \colon [0,1] \to X' \text{ such that } [\omega] \in \ \pi_1(X').$$

We call $A_1 = \operatorname{Int} c$ and $A_2 = X' - \{p\}$ where $p = e_c(0)$, then we can write $X' = A_1 \cup A_2$.
Observe that A_1 is path connected since it is the interior of a cell of dimension $i \geq 3$, while $c - \{0\}$ strongly deforms to $\operatorname{Bd} c$.
Hence, by applying the characteristic map e_c we have that

$$X' - \{p\} = X \cup_{f_c} (c - \{0\})$$

deforms to

$$X \cup_{f_c} \operatorname{Bd} c = X.$$

So, A_2 is path connected since so is S^{i-1}, being $i - 1 \geq 2$.
Moreover, $A_1 \cap A_2$ is homeomorphic to $\operatorname{Int} B^i - \{0\} \simeq S^{i-1}$, and hence $A_1 \cap A_2$ is path connected for $i \geq 2$.
Then, $\pi_1(A_1) \cong *$, $\pi_1(A_2) \cong \pi_1(X)$ and $\pi_1(A_1 \cap A_2) \cong \pi_1(S^{i-1}) \cong 0$.
Thus, we can apply Seifert Van Kampen theorem to get

$$\begin{aligned} \pi_1(X') &\cong \frac{\pi_1(A_1) * \pi_1(A_2)}{N(\{j_1(\omega), j_2^{-1}(\omega), \ \ \omega \in \pi_1(A_1 \cap A_2)\})} \cong \\ &\cong \frac{0 * \pi_1(X)}{0} = \pi_1(X). \end{aligned}$$

□

This lemma guarantees that while attaching cells of dimension greater than or equal to 3 the fundamental group is left unchanged.

Proposition 1.3.5. *Let X be a connected CW-complex of dimension k. Then the inclusion $X^{i-1} \subset X^i$ induces an isomorphism $\pi_1(X^{i-1}) \to \pi_1(X^i)$ for $i \geq 3$. Hence, we have*

$$\pi_1(X) \cong \pi_1(X^2).$$

Proof. By Lemma 1.3.4, we know that the fundamental group of a path connected space X is unaffected by attaching an i-cell of dimension $i \geq 3$. Hence, by induction on the number of cells of X of dimension $i \geq 3$, the proposition follows directly. □

1.4 Presentations of groups

A *presentation* of a group $\mathcal{G}$ consists of a set S of generators and a set R of relations such that any relation $\rho \in R$ is an element of the free group $F(S)$ and $\mathcal{G}$ is isomorphic to the quotient group $F(S)/N(R)$, where $N(R)$ is the normal subgroup generated by R. We denote a presentation of $\mathcal{G}$ by

$$\mathcal{G} = \langle S|R\rangle \cong \frac{F(S)}{N(R)}.$$

Determining whether two group presentations define isomorphic groups is undecidable.
However, given a presentation $\mathcal{G} = \langle \beta_1, \dots, \beta_n | \rho_1, \dots, \rho_k \rangle$ for $\mathcal{G}$, it is possible to obtain other presentations for $\mathcal{G}$ by applying the following operations called *Tietze transformations*.

i) *Adding a generator*: if α can be written in terms of $\beta_1, \dots, \beta_n$ as $\alpha = W(\beta_1, \dots, \beta_n)$, then we can insert α as an additional generator together with the relation $\alpha^{-1}W(\beta_1, \dots, \beta_n)$ and then we have the following new presentation for $\mathcal{G}$

$$\mathcal{G} = \langle \beta_1, \dots, \beta_n, \alpha | \rho_1, \dots, \rho_k, \alpha^{-1}W(\beta_1, \dots, \beta_n)\rangle.$$

ii) *Removing a generator*: if β_i can be written in terms of the other generators $\beta_1, \dots, \beta_{i-1}, \beta_{i+1}, \dots \beta_n$ as

$$\beta_i = W(\beta_1, \dots, \beta_{i-1}, \beta_{i+1}, \dots \beta_n),$$

then we can delete β_i and replace it by $W(\beta_1, \dots, \beta_{i-1}, \beta_{i+1}, \dots \beta_n)$ in the relations containing β_i.

iii) *Adding a relation*: if σ is a relation which can be derived from $\rho_1, \dots, \rho_k$ then we can insert σ and we have the following new presentation for $\mathcal{G}$

$$\mathcal{G} = \langle \beta_1, \dots, \beta_n | \rho_1, \dots, \rho_k, \sigma \rangle.$$

iv) *Removing a relation*: if ρ_i is a consequence of $\rho_1, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_k$ then we can delete ρ_i,

$$\mathcal{G} = \langle \beta_1, \dots, \beta_n | \rho_1, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_k \rangle.$$

It is possible to derive an alternative formulation of Seifert-Van Kampen Theorem in terms of group presentations.

Theorem 1.4.1. *(See [CF], Theorem 3.6)*
Let $\pi_1(X_1, x_0) \cong \langle \alpha_i | \lambda_j \rangle$, $\pi_1(X_2, x_0) \cong \langle \beta_i | \mu_j \rangle$ *and* $\pi_1(X_1 \cap X_2) \cong \langle \gamma_i | \nu_j \rangle$. *Then*

$$\pi_1(X_1 \cup X_2) \cong \langle \alpha_i,\ \beta_i | \lambda_j,\ \mu_j,\ j_1(\gamma_i) j_2(\gamma_i)^{-1} \rangle.$$

Consider a connected graph G, a maximal spanning tree T and a vertex v of T. Let $e_1, \dots, e_n$ be the edges of G not contained in T such that $F_G(e_i) = [v_i, w_i]$. We consider the paths g_i from v to v_i and h_i from w_i to v inside the tree T. Then we can take the classes $[f_i]$ of loops $f_i = g_i e_i h_i$.

Proposition 1.4.2. *The fundamental group* $\pi_1(G)$ *is the free group on the classes* $[f_1], \dots, [f_n]$.

Proof. It can be proved by induction on the number n of edges of G not contained in T.
For $n = 0$, G is a tree and hence $\pi_1(G) \cong 0$.
We assume that the thesis holds for $n - 1$ and we prove it for n edges out of T.
For each $i = 1, \dots, n$ we choose a point $x_i \in e_i$ and we consider

$$A_1 = G - \{x_1, \dots, x_{n-1}\} \quad \text{and} \quad A_2 = G - \{x_n\}.$$

Then, A_1 and A_2 are open, $A_1 \cap A_2 \simeq T$, $A_1 \simeq T \cup e_n$ and $A_2 \simeq G - e_n$. By inductive hypothesis, $\pi_1(A_2)$ is the free group on $[f_1], \dots, [f_{n-1}]$. Moreover, $\pi_1(A_1)$ is the free group on $[f_n]$ and $\pi_1(A_1 \cap A_2) \cong \pi_1(T) \cong 0$. Hence, by applying Seifert Van Kampen theorem, we get that $\pi_1(G)$ is the free group on $[f_1], \dots, [f_n]$. □

Proposition 1.4.3. *Suppose that* Γ *is a graph with a single vertex of degree greater than* 2, *then the fundamental group* $\pi_1(\Gamma)$ *is the free group on* $1 - \chi(\Gamma)$ *generators.*

Proof. First, notice that Prop. 1.4.2 means that a connected graph G is homotopy equivalent to a wedge of finitely many copies of S^1. Hence, in this case Γ is homotopy equivalent to a bouquet of loops. If we compute the Euler characteristic we get $\chi(\Gamma) = 1 - n$ where n indicates the number of loops constituting the bouquet. So, $\pi_1(\Gamma)$ is a free group on $n = 1 - \chi(\Gamma)$ generators. □

By Prop. 1.3.5, remind that a group presentation for $\pi_1(X^2)$ is also a group presentation for $\pi_1(X)$.

Lemma 1.4.4. *Let X be a path connected topological space and let us attach a 2-cell (c, h_c) to X via the attaching map $f_c\colon \operatorname{Bd} c \to X$.*
Suppose α is a generator for $\pi_1(\operatorname{Bd} c)$, if $\pi_1(X) = \langle \beta_1, \dots, \beta_n | \rho_1, \dots, \rho_n \rangle$ then,

$$\pi_1(X \cup_{f_c} c) = \langle \beta_1, \dots, \beta_n | \rho_1, \dots, \rho_n, \sigma \rangle$$

where σ is an expression for $f_{c*}(\alpha) \in \pi_1(X)$ written in terms of $\beta_1, \dots, \beta_n$.

Proof. Notice that $\pi_1(c) \cong 0$ and that by hypothesis, $\pi_1(X \cap c) \cong \pi_1(\operatorname{Bd} c)$. Hence, by applying Seifert-Van Kampen theorem in terms of group presentations, we have that a presentation for $\pi_1(X \cup_{f_c} c)$ is exactly $\langle \beta_1, \dots, \beta_n | \rho_1, \dots, \rho_n, \sigma \rangle$ where $\sigma = j_1(\alpha) j_2(\alpha)^{-1}$. □

Proposition 1.4.5. *Let X be a connected CW complex, $\beta_1, \dots, \beta_n$ the generators for the free group $\pi_1(X^1)$ as given in Prop. 1.4.2 and $(c_1, h_{c_1}), \dots, (c_k, h_{c_k})$ the 2-cells of X. For each $i = 1, \dots, k$, let α_i be any generator for $\pi_1(\operatorname{Bd} c_i) \cong \mathbb{Z}$ and σ_i be the expression of α_i in terms of $\beta_1, \dots, \beta_n$. Then,*

$$\pi_1(X) = \langle \beta_1, \dots, \beta_n | \sigma_1, \dots, \sigma_k \rangle.$$

Proof. Since X^1 is a graph, then by Prop. 1.4.2, the generators $\beta_1, \dots, \beta_n$ are given by the edges out of a maximal spanning tree T of X^1.
Remind that X^2 is obtained by attaching to X^1 all the 2-cells of X and also $\pi_1(X^2) \cong \pi_1(X)$ by Prop. 1.3.5.
Then, we prove the statement by induction on the number k of 2-cells of X. For $k = 0$, we have just $\pi_1(X^1) = \langle \beta_1, \dots, \beta_n \rangle$. We assume that the thesis holds for $k - 1$ and we verify it for k.
Let X^1_k indicate the set resulting from attaching k 2-cells to X^1, then

$$\pi_1(X^1_k) = \pi_1(X^1_{k-1} \cup_{f_{c_k}} c_k),$$

and by inductive hypothesis $\pi_1(X^1_{k-1}) = \langle \beta_1, \dots, \beta_n | \sigma_1, \dots, \sigma_{k-1} \rangle$ where $\sigma_1, \dots, \sigma_k$ are expressions for $\alpha_1, \dots, \alpha_{k-1}$ in terms of $\beta_1, \dots, \beta_n$. Then, by applying again Lemma 1.4.4 the thesis follows. □

In summary, the generators of $\pi_1(X^2)$ are the edges outside of a maximal spanning tree T of X^1 and the relations of $\pi_1(X^2)$ come from looking at the boundaries of the 2-cells of X^2 and writing them as words in terms of the generators.

1.5 Classical Morse theory

Classical Morse theory was developed in the 1920*s* and 1930*s* by the American mathematician Marston Morse [Mo] with the aim of deducing topological information about a differentiable manifold M by the study of a smooth real-valued function $f\colon M \to \mathbb{R}$ defined on M.

The basic idea consists in considering a differentiable manifold $M \subset \mathbb{R}^n$ and a collection of parallel hyperplanes and slicing M with these hyperplanes in order to extrapolate information by the variation of the shape of each single slice of M.

For example, let M be a torus and $f\colon M \to \mathbb{R}$ be the height function which associates to each point of M the corresponding height with respect to a plane V tangent to M. We denote by M^a the set of points x of M such that $f(x) \leq a$. We consider the points p, q, r, s in M as in Figure 1.5.1, then the following hold true:

1) if $a \leq f(p)$, then $M^a = \emptyset$;

2) if $f(p) < a < f(q)$, then M^a is homeomorphic to a 2-cell;

3) if $f(q) < a < f(r)$, then M^a is homeomorphic to a cylinder;

4) if $f(r) < a < f(s)$, then M^a is homeomorphic to a compact manifold of genus 1 having a circle as boundary;

5) if $f(s) < a$, then M^a is the full torus.

In general, given a smooth manifold M and a smooth function $f\colon M \to \mathbb{R}$, a point x in M is called a *critical* point of f if the induced map $f_*\colon T_xM \to T_{f(x)}\mathbb{R}$ is zero.

A critical point x is called *non-degenerate* if the Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j}(x))$ with respect to some local coordinates $x_1, \ldots, x_n$ is non-singular.

The function f is a *Morse function* if all its critical points are non-degenerate.

Lemma 1.5.1. *([Mi], Lemma 2.2) Given a smooth function $g\colon M \to \mathbb{R}$ and an $\varepsilon > 0$, there always exists a Morse function $f\colon M \to \mathbb{R}$ such that $|g(x) - f(x)| < \varepsilon$ for all $x \in M$.*

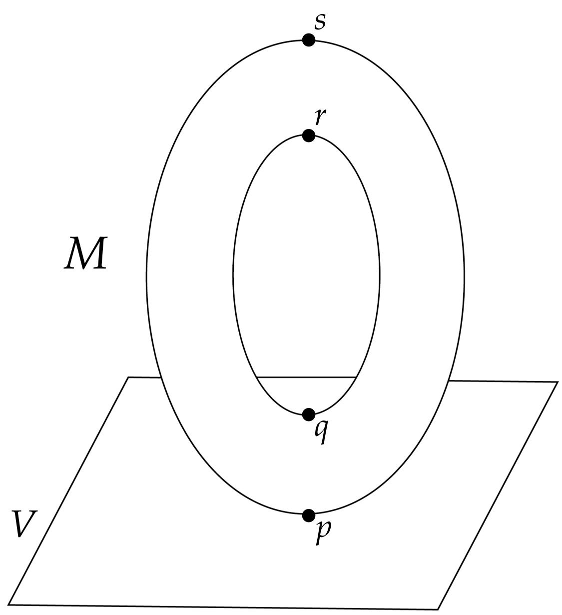

Figure 1.5.1: *Torus M tangent to a plane V at the point p.*

The *index* λ of a point p with respect to a smooth function f is the maximum dimension of a subspace of T_pM such that the Hessian of f is negative definite.

Two main results in classical Morse theory are the following:

Theorem 1.5.2. *([Mi], Theorem 3.1) Let $f\colon M \to \mathbb{R}$ be a smooth function and $a < b \in \mathbb{R}$. If there are no critical points in $f^{-1}([a,b])$, then M^a is diffeomorphic to M^b. Moreover, there exists a strong deformation of M^b onto M^a, and hence the inclusion map $i\colon M^a \to M^b$ is a homotopy equivalence.*

Theorem 1.5.3. *([Mi], Theorem 3.2) Let $f\colon M \to \mathbb{R}$ be a smooth function and p a non-degenerate critical point of f of index λ. Suppose $f(p) = c$, if p is the only critical point contained in $f^{-1}(c-\varepsilon, c+\varepsilon)$ for some ε, then, for ε sufficiently small, $M^{c+\varepsilon}$ is homotopy equivalent to $M^{c-\varepsilon}$ with a λ-cell attached.*

Observe that the points p, q, r, s in Figure 1.5.1 are non-degenerate critical points of f and the topology of M^a changes from one case to the following one by attaching a cell to M^a as soon as we cross a critical non-degenerate point.

In particular we can pass from 1) to 2) by attaching a 0-cell, from 2) to 3) and also from 3) to 4) by attaching a 1-cell, and finally from 4) to 5) by

attaching a 2-cell.
In this sense, classical Morse theory is able to extract topological issues about M from the study of a Morse function.

In general, as a consequence of the two theorems above, we have the following.

Theorem 1.5.4. *([Mi], Theorem 3.5) Let f be a smooth function on M with no degenerate critical points, if each M^a is compact, then M is homotopic equivalent to a CW-complex with one cell of dimension λ for each critical points of index λ.*

1.6 Discrete Morse theory

In 1995 another American mathematician, Robin Forman, published his first paper [Fo95] about a new adaptation of classical Morse theory he called discrete Morse theory.

The main difference from the original theory consists in considering CW complexes instead of manifolds and replacing smooth functions by discrete ones. The convenience in substituting the main object is reducing its complexity and at the same time not losing important information.

For the sake of simplicity, we focus on discrete Morse theory based on simplicial complexes, but all the following definitions can be adapted to the more general case of CW complexes.

Definition 1.6.1. Let K be a simplicial complex, a function $f\colon K \to \mathbb{R}$ is a *discrete Morse function* if, for every n-simplex $\sigma \in K$, the following are satisfied:

i) $U_\sigma = \#\{\tau > \sigma \mid f(\sigma) \geq f(\tau), \text{ where } \tau \text{ is an } (n+1)\text{-simplex }\} \leq 1;$

ii) $L_\sigma = \#\{\rho < \sigma \mid f(\rho) \geq f(\sigma), \text{ where } \rho \text{ is an } (n-1)\text{-simplex }\} \leq 1.$

Condition *i*) states that for each n-simplex σ in K, for all the simplices τ such that $\dim \sigma = \dim \tau - 1$, function f can associate to τ a real value smaller or equal to $f(\sigma)$ at most in one single case.
Similarly, condition *ii*) states that for each n-simplex σ in K and for all the simplices ρ such that $\dim \rho = \dim \sigma - 1$, function f can associate to ρ a real value greater or equal to $f(\sigma)$ at most in one single case .
It means that every time we are considering two simplices with those characteristics, a discrete Morse function f associates a greater value to the simplex with greater dimension a part from at most one exception.

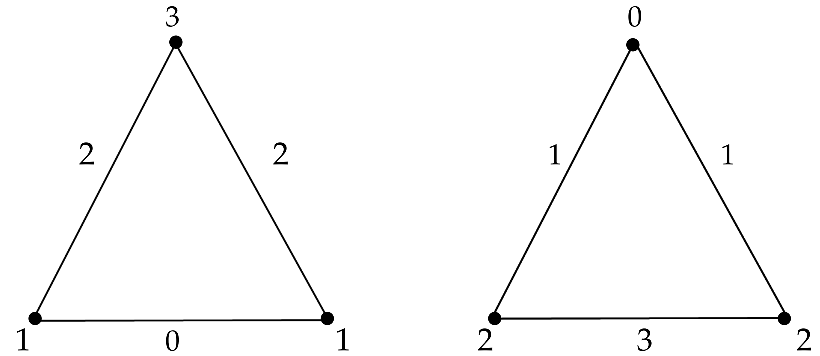

Figure 1.6.2

Example 1.6.2. In Figure 1.6.2 we have applied to the same 1-simplicial complex K, a function $f\colon K \to \mathbb{R}$ on the left and a function $f'\colon K \to \mathbb{R}$ on the right.

We can say that f is not a discrete Morse function because the edge $f^{-1}(0)$ violates condition *ii*) and the vertex $f^{-1}(3)$ violates condition *i*) in Def. 1.6.1.
On the other hand, f' is a Morse function since the edge ${f'}^{-1}(3)$ has greater value than both its vertices and the edges ${f'}^{-1}(1)$ have both one vertex with bigger value and one with smaller value, so both conditions *i*) and *ii*) in Def. 1.6.1 are satisfied.

For any simplicial complex K with a discrete Morse function f and $c \in \mathbb{R}$, we define the *level subcomplex* by

$$K(c) = \cup_{f(\tau)\leq c} \cup_{\sigma<\tau} \sigma$$

where σ, τ are simplices of K.

Definition 1.6.3. Given a simplicial complex K and a discrete Morse function $f\colon K \to \mathbb{R}$, an n-simplex σ in K is

i) *critical* if $L_\sigma = U_\sigma = 0$;

ii) *redundant* if $L_\sigma = 0$ and $U_\sigma = 1$;

iii) *collapsible* if $L_\sigma = 1$ and $U_\sigma = 0$.

Observe that Def. 1.6.3 means that a simplex σ is critical if the corresponding discrete Morse function f associates greater values to each simplex τ with higher dimension than σ and at the same time, associates smaller values to each simplex ρ with smaller dimension than σ.

Consider again the discrete Morse function f' in Figure1.6.2 on the right, we can observe now that the vertex ${f'}^{-1}(0)$ is a critical 0-simplex and edge ${f'}^{-1}(3)$ is a critical 1-simplex.

Let K_i be the set containing all the i-simplices, a *discrete vector field* is a map $W\colon K \to K \cup \{0\}$ such that

i) for each i, $W(K_i) \subseteq K_{i+1} \cup \{0\}$,

ii) for each i-simplex $\sigma \in K_i$, either $W(\sigma) = 0$ or σ is a regular face of $W(\sigma)$,

iii) if $\sigma \in \operatorname{Im} W$, then $W(\sigma) = 0$,

iv) for each i-simplex $\sigma \in K_i$, then $\#\{\rho \in K_{i-1} : W(\rho) = \sigma\} \leq 1$.

The following Theorems correspond respectively to Theorems 1.5.2, 1.5.3, 1.5.4 in classical Morse theory.

Theorem 1.6.4. *([Fo], Theorem 3.3) If there are no critical simplices σ with $f(\sigma) \in (a, b]$, then the level subcomplex $K(a)$ is a strong deformation of $K(b)$.*

Theorem 1.6.5. *([Fo], Theorem 3.4) If σ is the only critical n-simplex such that $f(\sigma) = c$ and $\sigma \in f^{-1}(c - \varepsilon, c + \varepsilon)$ for some $\varepsilon > 0$, then for all ε sufficiently small, $K(c + \varepsilon)$ is homotopy equivalent to $K(c - \varepsilon)$ with an n-cell attached.*

From the previous two lemmas next theorem follows easily:

Theorem 1.6.6. *([Fo], Corollary 3.5) Let K be a simplicial complex with a discrete Morse function $f\colon K \to \mathbb{R}$. Then, K is homotopy equivalent to a CW-complex with exactly one cell of dimension p for each critical simplex $\sigma \in K$ of dimension p.*

Chapter 2

Braid groups and their presentations

In this chapter we first introduce classical definitions of configuration spaces and braids and discuss how these two concepts are related and then, in the last section, we show some recent results concerning the problem of computing presentations for braid groups on graphs. Almost all of these statements will be proved again in the next chapter following a different approach.

2.1 Configuration spaces and braids

Let us give the definition of configuration space on a metric space as follows.

Definition 2.1.1. Let (X, d) be a metric space. The *labeled configuration space* of n points in X is

$$\widetilde{\mathcal{C}}_n(X) := \{(x_1, x_2, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

The *configuration space* $\mathcal{C}_n(X)$ of n points in X is the quotient of $\widetilde{\mathcal{C}}_n(X)$ by the natural action of the symmetric group

$$\mathcal{C}_n(X) := \widetilde{\mathcal{C}}_n(X)/\Sigma_n.$$

Given two configurations $x = \{x_0, \dots, x_n\}$ and $y = \{y_0, \dots, y_n\}$ in $\mathcal{C}_n(X)$, the distance between x and y is defined as

$$d_{\mathcal{C}_n(X)}(x, y) = \min_{\sigma \in \Sigma_n} \max_{i=1,\dots,n} d(x_i, y_{\sigma(i)}).$$

Let us prove that $d_{\mathcal{C}_n(X)}$ is a metric on $\mathcal{C}_n(X)$:

i) $d_{\mathcal{C}_n(X)}(x, y) \geq 0 \ \forall x, y \in \mathcal{C}_n(X)$ by non-negativity of the metric d.

ii) $d_{\mathcal{C}_n(X)}(x,y) = 0$ if and only if $\min_{\sigma\in\Sigma_n}\max_i d(x_i, y_{\sigma(i)}) = 0$ if and only if there exists a permutation $\sigma \in \Sigma_n$ such that $\max_i d(x_i, y_{\sigma(i)}) = 0$ if and only if $x_i = y_{\sigma(i)}$ for all i if and only if $x = y$.

iii)

$$d_{\mathcal{C}_n(X)}(x,y) = \min_{\sigma\in\Sigma_n}\max_i d(x_i, y_{\sigma(i)}) = \min_{\sigma\in\Sigma_n}\max_i d(x_{\sigma^{-1}(i)}, y_{\sigma(\sigma^{-1}(i))}) =$$
$$\min_{\sigma\in\Sigma_n}\max_i d(x_{\sigma^{-1}(i)}, y_i) = \min_{\sigma\in\Sigma_n}\max_i d(y_i, x_{\sigma^{-1}(i)}) =$$
$$\min_{\sigma\in\Sigma_n}\max_i d(y_i, x_{\sigma(i)}) = d_{\mathcal{C}_n(X)}(y,x).$$

iv)

$$d_{\mathcal{C}_n(X)}(x,z) + d_{\mathcal{C}_n(X)}(z,y) =$$
$$\min_{\sigma\in\Sigma_n}\max_i d(x_i, z_{\sigma(i)}) + \min_{\sigma'\in\Sigma_n}\max_{\sigma(i)} d(z_{\sigma(i)}, y_{\sigma'(\sigma(i))}) =$$
$$\min_{\sigma\in\Sigma_n}\max_i d(x_i, z_{\sigma(i)}) + \min_{\sigma'\sigma\in\Sigma_n}\max_{\sigma(i)} d(z_{\sigma(i)}, y_{\sigma'(\sigma(i))}) \geq$$
$$\min_{\sigma'\sigma\in\Sigma_n}\max_i d(x_i, y_{\sigma'(\sigma(i))}) = d_{\mathcal{C}_n(X)}(x,y). \quad \square$$

We define a corresponding metric $d_{\widetilde{\mathcal{C}}_n(X)}$ as follows

$$d_{\widetilde{\mathcal{C}}_n(X)}(x,y) = \max_{i=1,\dots,n} d_X(x_i, y_i).$$

Notice that the canonical projection $\pi\colon \widetilde{\mathcal{C}}_n(X) \to \mathcal{C}_n(X)$ is a regular covering. For any configuration $x = \{x_1, \dots, x_n\}$ in $\mathcal{C}_n(X)$, we take a $\delta_x > 0$ satisfying $\delta_x < \frac{1}{2}\min_{i,j} d(x_i, x_j)$. Then, $B(x,\delta) \subset \mathcal{C}_n(X)$ is well covered by π. Let $y \in \mathcal{C}_n(X)$ such that $y \in B(x,\delta)$, then if we restrict on each sheet of the covering, the metric $d_{\widetilde{\mathcal{C}}_n(X)}$ coincide with $d_{\mathcal{C}_n(X)}$.
Observe that, $\pi_{|B(x,\delta)}$ is invertible since it is an homeomorphism, and its inverse $\pi^{-1}_{|B^n}$ associates a numbering $(x_1, \dots, x_n)$ to $x = \{x_1, \dots, x_n\}$.

Notice that we can extend this numbering to all the points in $B(x,\delta)$. Hence, if $d_{\mathcal{C}_n(X)}(x,y) < \delta$, there is a numbering associated to y by $\pi^{-1}_{|B^n}$.
Moreover, $\pi^{-1}_{|B^n}$ is an isometry since X is locally isometric to $\pi^{-1}(X)$.

Fixed two base points $* \in \mathcal{C}_n(G)$ and $\tilde{*} \in \widetilde{\mathcal{C}}_n(G)$ such that $\pi(\tilde{*}) = *$, an *n-braid* B is a loop $\omega : [0,1] \to \mathcal{C}_n(X)$ based at $*$, while a *pure* n-braid P is

a loop $\omega : [0,1] \to \widetilde{\mathcal{C}}_n(X)$ based at $\tilde{*}$.
The set of n-braids equipped with the operation of concatenation of loops and quotiented out by the homotopy equivalence relation forms a group called the *braid group* and denoted by $\mathcal{B}_n(X)$. In other words, the braid group is the fundamental group of the configuration space of n points on X :

$$\mathcal{B}_n(X) = \pi_1(\mathcal{C}_n(X)).$$

Similarly, the *pure braid group* $\mathcal{P}_n(X)$ is the fundamental group of the labeled configuration space:

$$\mathcal{P}_n(X) = \pi_1(\widetilde{\mathcal{C}}_n(X)).$$

Up to the injective group homomorphism induced by the covering π, we have the inclusion $\mathcal{P}_n(X) \subset \mathcal{B}_n(X)$. Moreover, the path lifting property of π determine a group homomorphism $\sigma : \mathcal{B}_n(X) \to \Sigma_n$ such that $\mathcal{P}_n(X) = \ker \sigma$.

2.2 Classical braid groups

The theory of braids was classically developed for $X = \mathbb{R}^2$ starting from the seminal work of Artin [Ar]. Braids on $\mathbb{R}^2$ admit a geometrical interpretation in $\mathbb{R}^3$ as described below.
In $\mathbb{R}^3$, let us consider n points of coordinates $(i, 0, 0)$ and n points of coordinates $(\sigma_B(i), 0, 1)$ for $i = 1, \ldots, n$ and $\sigma_B \in \Sigma_n$.

An *n-braid* $B = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ is a disjoint union of n arcs A_i from $(i, 0, 0)$ to $(\sigma_B(i), 0, 1)$ such that the z-coordinate is monotonically increasing along each A_i.

Two braids B, B' are equivalent if there exists a continuous family of braids $(B_s)_{s\in[0,1]}$ such that $B_0 = B$ and $B_1 = B'$.

The composition of two braids B_1, B_2 is given by putting B_2 on the top of B_1 and then rescaling their union, that is

$$B_1B_2 = (1, 1, 1/2)(B_1 \cup (B_2 + (0, 0, 1))).$$

A presentation for the braid group $\mathcal{B}_n = \mathcal{B}_n(\mathbb{R}^2)$ was given by Artin [Ar], as follows:

$$\mathcal{B}_n = \langle\ \sigma_1, \ldots, \sigma_n \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } i + 1 < j\ \rangle.$$

Here, the generator σ_i represents the n-braid where the i-th strand crosses over the $(i + 1)$-th strand.

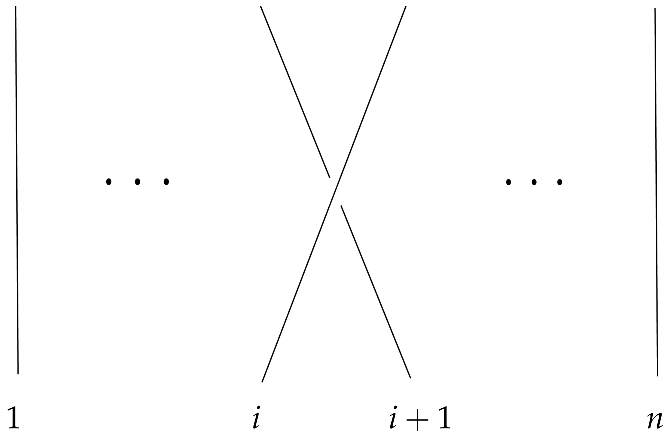

Figure 2.2.3: Diagram of the n-braid σ_i.

As a natural generalization of $\mathcal{B}_n$, an *Artin group* A is a group with a presentation of the following form:

$$A = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \dots}_{m_{ij}} = \underbrace{s_j s_i s_j \dots}_{m_{ji}} \text{ for } i \neq j \rangle$$

where $m_{ij} \geq 2$ is even and $m_{ij} = m_{ji}$ or $m_{ij} = \infty$.

A *right-angled* Artin group has a presentation in which $m_{ij} \in \{2, \infty\}$ for all i, j. It means that all the defining relations are commutators of the generators: $s_i s_j = s_j s_i$.

2.3 Braids on graphs

In the 1990's, some mathematicians started working on the problem of safe control schemes for automated guided vehicles (robots). The aim was finding the best way to let the robots move in their workspace avoiding collissions and at the same time guaranteeing a certain efficiency. The workspace was modelled by configuration spaces on manifolds but, in order to reduce the sophistication required for the production of these robots, it was imagined to let them move only on guidepath wires and so, those mathematicians started studying configuration spaces on graphs. In particular, we are going to see some results obtained by Robert Ghrist in [Gh99] and [Gh07], concerning configuration spaces on graphs and hence, concerning braids on graphs.

From now on $G = (V_G, E_G, F_G)$ will always denote a connected topological graph not homeomorphic to S^1 and for the sake of simplicity, we rename $V_G = V, E_G = E, F_G = F$.

Theorem 2.3.1. *([Gh99], Theorem 2.6 and 3.3) Let G be a connected graph and ℓ be the cardinality of $\{v \in V : \deg(v) > 2\}$. Then, the configuration spaces $\widetilde{\mathcal{C}}_n(G)$ and $\mathcal{C}_n(G)$ strongly deform on a complex of dimension at most ℓ.*

Sketch of proof. The result is first proved for trees and then generalized to any generic graph G by induction on the number of points n of $\mathcal{C}_n(G)$, the number of edges which are incident to a vertex separated from a terminal vertex by an edge and on the number of vertices with degree greater than 2.

Observe that G cannot be homeomorphic to S^1 indeed, in this case, G has no vertices of degree greater than 2, but both its configuration spaces $\widetilde{\mathcal{C}}_n(G)$ and $\mathcal{C}_n(G)$ strongly deform on S^1.

Corollary 2.3.2. *For any graph G with a single vertex v of degree $k > 2$, the configuration spaces $\mathcal{C}_n(G)$ and $\widetilde{\mathcal{C}}_n(G)$ strongly deform on a graph.*

The graphs with a single vertex of degree greater than 2 can be distinguished in the following families:

1) T_k with $k \geq 3$, the radial tree consisting of k terminal vertices $v_1, \dots, v_k$, a single vertex v_0 of degree k and k edges such that each edge e_i joins v_0 and v_i;

2) L_k with $k \geq 2$, the graph consisting of k loops all attached to a single vertex v_0 of degree $2k$;

3) $G_{k,h}$ with $k, h \geq 1$, the graph consisting of k terminal vertices $v_1, \dots, v_k$, a single vertex v_0 of degree greater than 2, h loops attached to v_0 and k edges such that each edge e_i joins v_0 and v_i for $i = 1, \dots, k$. In particular the degree of v_0 is equal to $2h + k$.

In these cases it is possible to determine the braid groups by computing the Euler characteristic of the graphs by Prop. 1.4.3.

Ghrist first in [Gh99] and then also in [Gh07], proved the following result.

Proposition 2.3.3. *Let T_k be the radial tree consisting of k edges (not loops) attached to a central vertex v_0 of degree k. Then, the pure braid group $\mathcal{P}_n T_k$ is isomorphic to a free group on*

$$1 + (nk + 1 - k - 2n)\frac{(n+k-2)!}{(k-1)!}$$

generators.

Proof. The Euler characteristic of $\mathcal{C}_n(T_k)$ is computed using a double induction on n and on k, fixing the point on the k-th edge of T_k which is the farthest from the central vertex v_0. So it can be obtained the following expression:

$$\chi(\mathcal{C}_n(T_k)) = \chi(\mathcal{C}_n(T_k)) + n\chi(\mathcal{C}_{n-1}(T_k)) - n\prod_{i=1}^{n-1}(k+i-2)$$

where the first term is due to the case where there are no points on the interior of the k-th edge, the second term is given by fixing one point on the k-th edge and the product is given by fixing one point on the central vertex v_0. □

In [Gh99], it was hinted that if we consider the configuration space $\mathcal{C}_n(T_k)$ we need to reduce $\chi(T_k)$ by a factor of $n!$.
This result was formalized and proved by Doig who constructed an explicit deformation retract of $\mathcal{C}_n(T_k)$.

Proposition 2.3.4. *([Do]) The braid group $\mathcal{B}_n T_k$ is a free group on*

$$1 + (nk + 1 - k - 2n)\frac{(n+k-2)!}{n!(k-1)!}$$

generators.

In [Gh99], Ghrist also gave the following:

Conjecture 2.3.5. *The braid group of any tree T is an Artin braid group.*

In 2000, Abrams disproved this conjecture and revised it to apply only on planar graphs in his Phd Thesis, which unfortunately, we did not manage to consult.
The following further result was then proved in 2004 by Connoly and Doig.

Proposition 2.3.6. *([CD]) The tree braid group $\mathcal{B}_n T$ is a right-angled Artin group if T is linear.*

Until talking of configuration spaces which can be deformed onto a graph, we can describe them easily, but in general we do not.
In order to classify the configuration spaces of generic graphs, Ghrist and Abrams have referred to the following approximation result.

Recall that any graph G is a 1-dimensional CW complex. Then, the n-fold product of G inherits a cubical structure from G in such a way that each cell is a product of n non necessarily distinct cells in G. But, as soon as we remove the diagonal, the space $\widetilde{\mathcal{C}}_n(G)$ does not have a cell structure anymore. Anyway, it is possible to lead back to a convenient approximation of the whole configuration space.

Let $\Delta = \{(x_1, x_2, \dots, x_n) \in G^n \mid x_i = x_j \text{ for } i \neq j\}$ be the diagonal of G^n and Δ' denote the union of the open cells in G^n whose closures intersect Δ.

The *discretized configuration space* of n points on G is the maximum subcomplex of $\widetilde{\mathcal{C}}_n(G)$ which does not intersect Δ, and we denote it by $\widetilde{\mathcal{D}}_n(G) = G^n - \Delta'$.
Hence, any k-cell in $\widetilde{\mathcal{D}}_n(G)$ has the form $\bar{c}_1 \times \cdots \times \bar{c}_k$, where each c_i is a cell of G and $\bar{c}_i \cap \bar{c}_j = \emptyset$ for $i \neq j$.

The *unlabeled discretized configuration space* $\mathcal{D}_n(G)$ is the quotient of $\widetilde{\mathcal{D}}_n(G)$ by the action of Σ_n.
In his thesis, Abrams proved the following theorem which later was proved again also by Prue and Scrimshaw in [PS].

Theorem 2.3.7. *(Subdivision theorem)*
For $n \geq 2$, let G be any graph with at least n vertices. Then the configuration space $\widetilde{\mathcal{C}}_n(G)$ (respectively $\mathcal{C}_n(G)$) strongly deforms onto the discretized space $\widetilde{\mathcal{D}}_n(G)$ (respectively $\mathcal{D}_n(G)$) if G is sufficiently subdivided, in the following sense:

i) each path between distinct essential vertices has at least $n-1$ edges;

ii) each cycle containing at least one essential vertex has at least $n+1$ edges.

The subdivision theorem implies that if a graph G is sufficiently subdivided, $\widetilde{\mathcal{C}}_n(G)$ (respectively $\mathcal{C}_n(G)$) is homotopy equivalent to $\widetilde{\mathcal{D}}_n(G)$ (respectively $\mathcal{D}_n(G)$).

Sketch of the proof in [PS]. First it is proved that assuming G sufficiently subdivided, then there exists a CW-structure on $\mathcal{C}_n(G)$ such that the inclusion map $i\colon \mathcal{D}_n(G) \to \mathcal{C}_n(G)$ is a cellular map, i.e. sends i-skeleta in i-skeleta for all i. Then it is proved that the inclusion map i induces isomorphisms on all homotopy groups.

In 2005, Farley and Sabalka proved Theor. 2.3.1 by using the Subdivision theorem and the discrete Morse theory in the following way.
First we have the case of a tree T.
Let us denote by $\mathcal{D}_n(T)^k_r$ the k-skeleton of $\mathcal{D}_n(T)$ with the redundant k-cells removed and by $\mathcal{D}_n(T)^k_{r,c}$ the k-skeleton of $\mathcal{D}_n(T)$ with the redundant and the critical k-cells removed.

Theorem 2.3.8. *([FS05], Theorem 4.3) Let T be a tree and c a critical cell of $\mathcal{D}_n(T)$. Let $k = \min\{\lfloor \frac{n}{2} \rfloor, \#\{v \in T^0, \deg(v) > 2\}$. Then $\dim c \leq k$ and so $\mathcal{D}_n(T)$ strongly deforms on $\mathcal{D}_n(T)^k_r$.*

Sketch of proof. It can be proved that in c there are at least as many vertices as edges. Since the dimension of c is equal to the number of edges in c and the total number of cells in c is n, then $\dim c \leq \frac{n}{2}$.
By definition of critical cell follows also the other bound for $\dim c$.
Then, $\mathcal{D}_n(T)$ has no critical cells of dim $\geq k$ and it can be proved that there is an isomorphism between $\pi_{k-1}(\mathcal{D}_n(T)^k_r)$ and $\pi_k(\mathcal{D}_n(T)^k_{r,c})$. Hence, $\mathcal{D}_n(T)$ strongly deforms on $\mathcal{D}_n(T)^k_r$.

Now we see the case of a generic graph G.

Theorem 2.3.9. *([FS05], Theorem 4.4) Let G be a sufficiently subdivided graph and $\chi(G)$ its Euler characteristic. Then, $\mathcal{D}_n(G)$ strongly deforms to a CW-complex of dimension at most k, where*

$$k = \min\Big\{\lfloor \frac{n+1-\chi(G)}{2} \rfloor, \#\{v \in G^0 : \deg(v) > 2\}\Big\}.$$

Sketch of proof. First construct a maximal subtree of G such that all the edges out of T neighbor the vertices of G of degree $k > 2$. Then, any embedding of T in the plane induces a discrete gradient vector field such that every edge in a critical cell contains a vertex of degree $k > 2$. Hence, $\dim \mathcal{D}_n(G) \leq k$ since the dimension of the critical cells of $\mathcal{D}_n(G)$ is less than or equal to k with respect to the gradient vector field.
Then, observe that the number of edges out of T is equal to $1 - \chi(G)$ and the dimension of any critical cell of $\mathcal{D}_n(G)$ is bounded by

$$1 - \chi(G) + \lfloor \frac{n-1+\chi(G)}{2} \rfloor.$$

By using again the discrete Morse theory, they proved the following result regarding presentations for fundamental groups of CW-complexes, which they used then to compute presentations for the braid groups of trees.

Theorem 2.3.10. *([FS05], Theorem 2.5) Let X be a connected CW-complex with a discrete gradient vector field W. Let T be the maximal tree of X consisting of all the collapsible edges in X and additional critical edges if necessary. Then,*

$$\pi_1(X) \cong \langle S|R\rangle$$

where S is the set of positive critical 1-cells not contained in T and R is the set of certain reduced forms of the boundary of critical 2-cells.

Moreover, the result in Prop. 2.3.6 was widened as follows.

Theorem 2.3.11. *([FS08]) The tree braid group $\mathcal{B}_n T$ is a right-angled Artin group if and only if T is linear or $n < 4$.*

Sketch of proof. ($\Leftarrow$) By Prop. 2.3.6, if T is a linear tree then $\mathcal{B}_n T$ is a right-angled Artin group. If T is a tree and $n < 4$, it follows by Theorem 2.3.8.
($\Rightarrow$) It can be proved by contradiction, assuming that T is non linear, $n \geq 4$ and $\mathcal{B}_n T$ is a right-angled Artin group and then referring to the cohomology rings and the critical cells of $\mathcal{C}_n(T)$.

Chapter 3

Presentations of braid groups on graphs via cubical complexes

In this chapter, we want to reduce the problem of computing a presentation for the braid group of graphs to just computing a presentation for the fundamental group of a cubical complex, and hence by Prop. 1.4.5 it is sufficient to study the 1-cells and the 2-cells of the cubical complex.

First, we define a subspace $\mathcal{N}_n(G)$ of $\mathcal{C}_n(G)$ consisting of a kind of "normalized" configurations such that, for each edge e of the graph G, the first and the last point on e of the configuration are inside of two proper intervals and the intermediate points are uniformly distributed between them.

In the second section, we define a continuous mapping

$$\Phi\colon \mathcal{C}_n(G) \;\to\; \mathcal{C}_n(G)$$

such that $\operatorname{Im}\Phi = \mathcal{N}_n(G)$ and we prove that there exists a weak deformation of $\mathcal{C}_n(G)$ into $\mathcal{N}_n(G)$ by using Φ.

In the last section we construct a cubical complex $\mathcal{Q}_n(G)$ homeomorphic to $\mathcal{N}_n(G)$ and we derive that the braid group on G is isomorphic to the fundamental group of $\mathcal{Q}_n(G)$. Hence, we are able now to compute a presentation for the braid group by looking at the 1-cells and 2-cells of $\mathcal{Q}_n(G)$. Finally, in Prop. 3.3.5 we reduce the cubical complex $\mathcal{Q}_n(G)$ to a subcomplex which is still homotopic equivalent to $\mathcal{Q}_n(G)$ in order to simplify the computing.

3.1 The normalized configuration space $\mathcal{N}_n(G)$

Let $G = (V, E, F)$ be a connected graph not homeomorphic to S^1 and $\mathcal{C}(G) = \bigcup_{n\geq 0} \mathcal{C}_n(G)$ be the set of all finite configurations in G.

Given a configuration x in $\mathcal{C}(G)$, we put $x_V = x \cap V$ and similarly, for

each $e \in E$, we put $x_e = x \cap e^\circ$ the set of points of x contained in the interior of the edge e.
Notice that x is the disjoint union of x_V and the x_e's.

Let $\nu : \mathcal{C}(G) \to \{0,1\}^V \times \mathbb{N}^E$ be the mapping sending a configuration $x \in \mathcal{C}(G)$ to a couple $n_{V,E}(x) = ((n_v(x))_{v\in V}, (n_e(x))_{e\in E})$ where

$$n_v(x) = \begin{cases} 1 & \text{if } v \in x \\ 0 & \text{otherwise} \end{cases}$$

and $n_e(x) = |x_e|$. We also put $n_V(x) = \sum_{v\in V} n_v(x) = |x_V|$.

Definition 3.1.1. Let $\mathcal{C}_{red}(G) \subset \mathcal{C}(G)$ be the subset of the configurations $x \in \mathcal{C}(G)$ such that $n_e(x) \leq 2$ for each edge e.

Consider an edge $e \in E$ between the vertices $v, w \in V$ and take the unique isometry $\alpha_{e,v,w} \colon [-\frac{1}{2}, \frac{1}{2}] \to e$ such that $\alpha_{e,v,w}(-\frac{1}{2}) = v$ and $\alpha_{e,v,w}(\frac{1}{2}) = w$.

If $x_e \neq \emptyset$, then we can write it as $x_e = \{\alpha_{e,v,w}(t_1), \ldots, \alpha_{e,v,w}(t_{n_e(x)})\}$ such that $-\frac{1}{2} < t_1 < \cdots < t_{n_e(x)} < \frac{1}{2}$, where t_i is the coordinate of the i-th point of x_e according to the parametrization $\alpha_{e,v,w}$.
Let us call

$$\begin{aligned} t_{e,v} &= t_1 \\ t_{e,w} &= t_{n_e(x)}. \end{aligned} \tag{3.1.1}$$

Remark 3.1.2. Observe that if $v = w$, both t_1 and $t_{n_e(x)}$ are denoted by $t_{e,v}$. Then, if we consider the opposite parametrization $\alpha_{e,w,v} \colon [-\frac{1}{2}, \frac{1}{2}] \to e$ such that $\alpha_{e,w,v}(t) = \alpha_{e,v,w}(-t)$, we still get the same two points denoted again by $t_{e,v}$.
If $v \neq w$, and we consider the opposite parametrization $\alpha_{e,w,v}$, we get for $t_{e,v}$ and $t_{e,w}$ the same two values found using $\alpha_{e,v,w}$ but the notations $t_{e,v}$ and $t_{e,w}$ are swapped.

Let us set:

$$\begin{aligned} \bar{t}_{e,v} &= -\frac{1}{2}\frac{n_e(x)-1}{n_e(x)+1} \\ \bar{t}_{e,w} &= \frac{1}{2}\frac{n_e(x)-1}{n_e(x)+1} \end{aligned} \tag{3.1.2}$$

We call $I_{e,v} = [-\frac{1}{2}, \bar{t}_{e,v}]$ and $I_{e,w} = [\,\bar{t}_{e,w}, \frac{1}{2}]$ the *approaching interval* respectively to v and to w with respect to e.

Remark 3.1.3. These definitions are independent on the choice of the parametrization. Indeed, if we consider parametrization $\alpha_{e,w,v}$, then we still get the same coordinates $\bar{t}_{e,v}$ and $\bar{t}_{e,w}$ with v and w swapped if $v \neq w$.

Definition 3.1.4. Let G be a connected graph. The *normalized configuration space* $\mathcal{N}_n(G)$ of n points on G is a subspace of $\mathcal{C}_n(G)$ consisting of the configuration $x \in \mathcal{N}_n(G)$ which satisfies the properties below.
For each edge $e \in E$ with endpoints v and w, consider $x_e = \{x_1 = \alpha_{e,v,w}(t_1), \ldots, x_{n_e(x)} = \alpha_{e,v,w}(t_{n_e(x)})\}$ such that $-\frac{1}{2} < t_1 < \cdots < t_{n_e(x)} < \frac{1}{2}$. Then,

i) $-\frac{1}{2} < t_1 \leq \bar{t}_{e,v}$ and $\bar{t}_{e,w} \leq t_{n_e(x)} < \frac{1}{2}$.

ii) $t_{e,v} = \bar{t}_{e,v}$ if $t_{e',v} < \bar{t}_{e',v}$ for some edge $e' \neq e$ such that $v \in e'$ parametrized by $\alpha_{e',v,w'}$ and similarly $t_{e,w} = \bar{t}_{e,w}$ if $t_{e',w} > \bar{t}_{e',w}$ for some edge $e' \neq e$ such that $w \in e'$ parametrized by $\alpha_{e',v',w}$.

iii) $t_2, \ldots, t_{n_e(x)-1}$ are uniformly distributed between t_1 and $t_{n_e(x)}$ which means that the subintervals $[t_i, t_{i+1}]$ for $i = 2, \ldots, n_e(x) - 2$ have all the same amplitude.

3.2 The weak deformation of $\mathcal{C}_n(G)$ into $\mathcal{N}_n(G)$

Now we define a continuous function $\Phi\colon \mathcal{C}_n(G) \to \mathcal{C}_n(G)$ such that $\operatorname{Im}\Phi = \mathcal{N}_n(G)$ and then we use it to provide a weak deformation of $\mathcal{C}_n(G)$ into $\mathcal{N}_n(G)$.

Given an edge $e \in E$ of vertices $v, w \in V$, we define the following parameters which indicate the distances of $t_{e,v}$ and $t_{e,w}$ respectively from the vertices v and w divided by the amplitude of the approaching intervals $I_{e,v}, I_{e,w}$:

$$\delta_{e,v} = \begin{cases} 1 & \text{if } x_e = \emptyset \\ \min(1, (1/2 + t_{e,v})/(1/2 + \bar{t}_{e,v})) & \text{otherwise} \end{cases}$$

$$\delta_{e,w} = \begin{cases} 1 & \text{if } x_e = \emptyset \\ \min(1, (1/2 - t_{e,w})/(1/2 - \bar{t}_{e,w})) & \text{otherwise} \end{cases}$$

Then we define the *approaching parameters* $d_{e,v}$ of v and $d_{e,w}$ of w with respect to the edge e :

$$d_{e,v} = \begin{cases} 0 & \text{if } n_v(x) = 1 \\ 1 & \text{if } deg(v) = 1 \\ \min\left\{\delta_{e',v},\ e' \neq e \text{ s.t. } v \in e'\right\} & \text{otherwise,} \end{cases}$$

$$d_{e,w} = \begin{cases} 0 & \text{if } n_w(x) = 1 \\ 1 & \text{if } deg(w) = 1 \\ \min\left\{\delta_{e',w},\ e' \neq e \text{ s.t. } w \in e'\right\} & \text{otherwise.} \end{cases}$$

Finally, we put

$$r_{e,v}(x) = \begin{cases} 1 & \text{if } d_{e,v} = 0 \\ \min(1,\ \delta_{e,v}/d_{e,v}) & \text{otherwise} \end{cases}$$

$$r_{e,w}(x) = \begin{cases} 1 & \text{if } d_{e,w} = 0 \\ \min(1,\ \delta_{e,w}/d_{e,w}) & \text{otherwise.} \end{cases}$$

Then, we construct a function $\varphi : \mathcal{C}(G) \to \mathcal{C}_{red}(G)$ sending a configuration $x \in \mathcal{C}(G)$ to a reduced configuration $\varphi(x)$ such that:

a) if $n_e(x) = 0$, then $\varphi(x)_e = \emptyset$.

b) if $n_e(x) = 1$, that is x_e consists of a single point $x_1 \in x$, $t_{e,v} = t_{e,w} = t_1$ and $\bar{t}_{e,v} = \bar{t}_{e,w} = 0$, then $\varphi(x)_e$ consists of a single point such that:

$$\varphi(x)_e = \begin{cases} \alpha_{e,v,w}\Big(-\dfrac{1}{2} + \dfrac{r_{e,v}(x)}{r_{e,v}(x) + r_{e,w}(x)}\Big) & \text{if } -\frac{1}{2} < t_1 < 0 \\ \alpha_{e,v,w}\Big(\dfrac{1}{2} - \dfrac{r_{e,w}(x)}{r_{e,v}(x) + r_{e,w}(x)}\Big) & \text{if } 0 < t_1 < \frac{1}{2}. \end{cases}$$

c) If $n_e(x) \geq 2$, then

$$\begin{aligned} \varphi(x)_e = \Big\{&\alpha_{e,v,w}\Big(-\frac{1}{2} + \frac{r_{e,v}(x)}{r_{e,v}(x) + r_{e,w}(x) + n_e(x) - 1}\Big), \\ &\alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(x)}{r_{e,v}(x) + r_{e,w}(x) + n_e(x) - 1}\Big)\Big\}. \end{aligned} \tag{3.2.3}$$

This means that for each edge $e \in E$, φ associates to x_e the configuration $\varphi(x)_e$ in (3.2.3) consisting of 2 points which depends on the coordinates

$t_{e,v}$ and $t_{e,w}$ and on the values of the approaching parameters $d_{e,v}, d_{e,w}$. Thus, the number of points constituting a configuration $\varphi(x) \in \mathcal{C}_{red}(G)$ is $n_{red}(x) = \sum_{e \in E} \min(2, n_e(x)) + n_V(x)$.

Now define a function $\rho \colon \mathcal{C}_{red}(G) \times (\{0,1\}^V \times \mathbb{N}^E) \to \mathcal{C}(G)$ such that

$$\rho(x, n_{V,E})_V = \{v \in V : n_v(x) = 1\} = x_V$$

and moreover,

i) if $n_e \leq 2$, then ρ keeps the points of x_e, that is $\rho(x, n_{V,E})_e = x_e$;

ii) if $n_e > 2$, then ρ adds $n_e - 2$ points on the edge e whose coordinates are uniformly distributed between $t_{e,v}$ and $t_{e,w}$.
In this case, we write ρ as follows:

$$\rho(x, n_{V,E})_e = \Big\{\alpha_{e,v,w}\Big(\Big(1 - \frac{k}{n_e - 1}\Big)t_{e,v} + \frac{k}{n_e - 1}t_{e,w}\Big) \text{ for } k = 0, \cdots, n_e - 1\Big\}.$$

Notice that $x \subset \rho(x, n_{V,E})$, i.e. ρ keeps each configuration $x \in \mathcal{C}_{red}(G)$. Moreover, observe that for any configuration $x \in \mathcal{C}_{red}(G)$, the configuration $\rho(x, n_{V,E})$ is obtained by convex combinations of points of x. Hence, ρ is always continuous.

Finally, we define

$$\Phi \colon \mathcal{C}(G) \to \mathcal{C}(G)$$

such that

$$\Phi = \rho \circ (\varphi \times \nu).$$

Proposition 3.2.1. *For any configuration $x \in \mathcal{C}_n(G)$ and any numbering $\tilde{x} = (x_1, \ldots, x_n) \in \widetilde{\mathcal{C}}_n(G)$ there exists a $\overline{\delta} > 0$ sufficiently small such that, for any $0 < \delta < \overline{\delta}$ and for any $y \in \mathcal{C}_n(G)$ with $d_{\mathcal{C}_n(G)}(x, y) < \delta$, there exists a unique numbering $\tilde{y} = (y_1, \ldots, y_n)$ of y such that:*

i) for all $i = 1, \ldots, n$, x_i and y_i are the unique points respectively of x and y such that $d_G(x_i, y_i) < \delta$, and so

$$d_{\mathcal{C}_n(G)}(x, y) = \max_{i=1,\ldots,n} d_G(x_i, y_i).$$

ii) If $x_e = (x_{i_1}, \ldots, x_{i_k})$ then also $y_e = (y_{i_1}, \ldots, y_{i_k})$ where $k = n_e(x) = n_e(y)$ and the two numberings respect the same order along e.

iii) If x_i is a vertex v, then y_i is the first point of y along the edge e which minimizes $\delta_{e,v}$ among all the edges having v as an endpoint.

Proof. We assume that δ satisfies the following inequalities:

$$\delta < \frac{1}{2} \min \Big\{ d_G(x_i, x_j), \text{ for all } i \neq j \Big\}, \tag{3.2.4}$$

$$\delta < \frac{1}{2} \min \Big\{ d_G(x_i, v), \, x_i \neq v, \, v \in V \Big\} \tag{3.2.5}$$

and

$$\delta < \frac{1}{4 \max_{e \in E}(n_e(x) + 2)} \min_{e,v} \left(\frac{1}{2} + t_{e,v} \right) \tag{3.2.6}$$

By (3.2.4) we have that for each point x_i of $\tilde{x}$ there exists a unique point of y which we denote by the same index y_i such that $d_G(x_i, y_i) < \delta$. We put $\tilde{y} = (y_1, \ldots, y_n)$. Then, by definition of the metric $d_{\mathcal{C}_n(G)}$, it follows that $d_{\mathcal{C}_n(G)}(x, y) = \max_{i=1,\ldots,n} d_G(x_i, y_i)$. So condition i) is satisfied.

By (3.2.5) we have that $n_e(x) = n_e(y)$ indeed if x_{i_1} is the point of x_e of minimum distance from v, then y_{i_1} is contained on the same edge e of x_i. Then, using again (3.2.4), we guarantee that the renumberings of x_e and y_e respect the same order along e. Hence condition ii) is satisfied.

Finally, condition iii) is satisfied, indeed if $x_i = v$ then (3.2.6) guarantees that there is a single point of y minimizing $\delta_{e,v}$ and moreover, by (3.2.4), it has the same index i of x_i. □

Proposition 3.2.2. *Φ is a continuous mapping.*

Proof. First remind that function Φ is defined in such a way that for each configuration $x \in \mathcal{C}_n(G)$ then $\Phi(x)_V = x_V$ and there is a bijection between x_e and $\Phi(x)_e$ such that the order of numbering along e is preserved.

Consider the covering projection $\pi \colon \widetilde{\mathcal{C}}_n(G) \to \mathcal{C}_n(G)$ which sends a configuration $\tilde{x}$ in $\widetilde{\mathcal{C}}_n(G)$ to a configuration $x \in \mathcal{C}_n(G)$. Then, we can construct a function $\widetilde{\Phi} \colon \widetilde{\mathcal{C}}(G) \to \widetilde{\mathcal{C}}(G)$ such that $\widetilde{\Phi}(\tilde{x}) = \Phi(x)$ where $\widetilde{\Phi}(\tilde{x})$ inherites the order of numbering along each edge $e \in E$ by the configuration $\tilde{x}$. In particular, $\pi \circ \widetilde{\Phi} = \Phi \circ \pi$.

Notice that in order to prove continuity of Φ, it is sufficient to prove the continuity of $\widetilde{\Phi}$, indeed Φ is locally given by $\pi \circ \widetilde{\Phi} \circ \pi_{|}^{-1}$ where $\pi_{|}^{-1}$ is a local inverse of π defined on a well covered open set.

Moreover, since $\widetilde{\mathcal{C}}(G) \subset \bigcup_n \prod_n G$, it is sufficient to verify the continuity at each configuration $\tilde{x} \in \widetilde{\mathcal{C}}_n(G)$ of each single component $\widetilde{\Phi}_i$ of $\widetilde{\Phi}$.

Since the function ρ is continuous, as we said above, it is enough to consider the cases when $\widetilde{\Phi}(\tilde{x})_i$ is a vertex $w \in V$ or $\widetilde{\Phi}(\tilde{x})_i$ is the first or the last point of $\widetilde{\Phi}(x)$ along an edge e. In the latter case, we verify just when $\widetilde{\Phi}(\tilde{x})_i$ is the last point $\widetilde{\Phi}(x)_{e,w}$ of $\widetilde{\Phi}(\tilde{x})$ along an edge e, indeed the proof is independent on the choice of the parametrization.

Given a configuration $\tilde{x} \in \widetilde{\mathcal{C}}_n(G)$, let $\tilde{y}$ be a configuration in $\widetilde{\mathcal{C}}_n(G)$, satisfying the conditions described in Prop. 3.2.1.

Case A If $\widetilde{\Phi}(\tilde{x})_i$ is a vertex $w \in V$ then, $\tilde{x}_i = w$ and there are the following possibilities for $\tilde{y}_i$.

1) $\tilde{y}_i = w$, then also $\widetilde{\Phi}(\tilde{y})_i = w$. Hence, $\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i = \widetilde{\Phi}(\tilde{x})_i$.

2) $\tilde{y}_i$ is the last point $y_{e,w}$ of y for some edge e between the vertices v and w, such that $\delta_{e,w} < \delta_{e',w}$ for all the edges $e' \neq e$ which have w as an endpoint.
If $v \in x$, then $v = \tilde{x}_j$ for some j and we have the following possibilitis for $\tilde{y}_j$.

2.1) $\tilde{y}_j$ is the vertex v or the first point $y_{e'',v}$ for some edge e'' such that $\delta_{e'',v} < \delta_{e,v}$. In this case, notice that $n_e(y) = n_e(x) + 1$, $r_{e,v}(x) = r_{e,w}(x) = r_{e,v}(y) = 1$ while $r_{e,w}(y)$ goes to zero as $\tilde{y}$ tends to $\tilde{x}$. Then,

$$\begin{aligned}
\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y) + r_{e,w}(y) + n_e(y) - 1}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}$$

2.2) $\tilde{y}_j$ is the first point $y_{e,v}$ along e, then $n_e(y) = n_e(x) + 2$, $r_{e,v}(x) = r_{e,w}(x) = 1$ while $r_{e,v}(y)$ and $r_{e,w}(y)$ go both to zero as $\tilde{y}$ tends to $\tilde{x}$. Hence,

$$\begin{aligned}\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\ &= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\ &= \alpha_{e,v,w}\Big(\frac{1}{2}\Big) = \widetilde{\Phi}(\tilde{x})_i.\end{aligned}$$

If $v \notin x$, then let j be such that $\tilde{x}_j$ is the first point $x_{e,v}$ of x along e and then, $\tilde{y}_j$ is the first point $y_{e,v}$ of y along e. Then, $n_e(y) = n_e(x)+1$, $r_{e,w}(x) = 1$ while $r_{e,w}(y)$ goes to zero as $\tilde{y}$ tends to $\tilde{x}$. Then,

$$\begin{aligned}\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\ &= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\ &= \alpha_{e,v,w}\Big(\frac{1}{2}\Big) = \widetilde{\Phi}(\tilde{x})_i.\end{aligned}$$

Case B If $\widetilde{\Phi}(\tilde{x})_i$ is the last point $\Phi(x)_{e,w}$ of $\Phi(x)$ along an edge e between the vertices v and w then, $\tilde{x}_i$ is the last point $x_{e,w}$ of x along e and we need to discuss the cases below.

1) If $w \in x$, then $w = \widetilde{\Phi}(\tilde{x})_j$ for some $j \neq i$ and we have three possible cases for $\tilde{y}_j$ as follows.

1.1) If $\tilde{y}_j = w$, then $\tilde{y}_i$ is the last point $y_{e,w}$ of y along e.

If $v \in x$, then $v = \tilde{x}_k$ for some k and in this case we have the following possibilities for $\tilde{y}_k$.

1.1.1) $\tilde{y}_k$ is the vertex v or the first point $y_{e',v}$ for some edge e' such that $\delta_{e',v} < \delta_{e,v}$. In this case, notice that $n_e(y) = n_e(x)$, $r_{e,v}(x) = r_{e,w}(x) = 1$ and also $r_{e,v}(y) = r_{e,w}(y) = 1$. We have,

$$\begin{aligned}
\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{1+n_e(x)}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(x)}{r_{e,v}(x)+r_{e,w}(x)+n_e(x)-1}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}$$

1.1.2) $\tilde{y}_k$ is the first point $y_{e,v}$ along the edge e. Notice that $r_{e,v}(x) = r_{e,w}(x) = 1$ and $r_{e,w}(y) = 1$, $n_e(y) = n_e(x) + 1$. So, we get

$$\begin{aligned}
\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{r_{e,v}(y)+n_e(y)}\Big) = \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{n_e(y)}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{n_e(x)+1}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}$$

If $v \notin x$, then let k be such that $\tilde{x}_k$ is the first point $x_{e,v}$ of x along e and then, $\tilde{y}_k$ is the first point $y_{e,v}$ of y along e and $n_e(y) = n_e(x)$.
Notice that $r_{e,w}(x) = r_{e,w}(y) = 1$ and then,

$$\begin{aligned}
\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{r_{e,v}(y)+n_e(y)}\Big) = \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{r_{e,v}(x)+n_e(x)}\Big) \\
&= \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}$$

1.2) Assume $\tilde{y}_j$ is the last point $y_{e,w}$ of y along e such that $\delta_{e,w}(y) < \delta_{e',w}(y)$ for all the edges e' having w as an endpoint.
If $v \in x$, then $v = \tilde{x}_k$ for some k and in this case we have the following possibilities for $\tilde{y}_k$.

1.2.1) $\tilde{y}_k$ is the vertex v or the first point $y_{e',v}$ for some edge e' such that $\delta_{e',v} < \delta_{e,v}$. In this case, notice that $n_e(y) = n_e(x) + 1$, $r_{e,v}(x) = r_{e,w}(x) = 1$ and also $r_{e,v}(y) = 1$. Then,

$$\begin{aligned}
&\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i = \\
&\lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\Big(1 - \frac{n_e(y)-2}{n_e(y)-1}\Big)\Big(-\frac{1}{2} + \frac{r_{e,v}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\
&+ \frac{n_e(y)-2}{n_e(y)-1}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big)\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{n_e(y)-1}\Big(-\frac{1}{2} + \frac{1}{n_e(y)}\Big) + \frac{1}{2}\frac{n_e(y)-2}{n_e(y)-1}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{(n_e(y)-2)(n_e(y)-1)}{2n_e(y)(n_e(y)-1)}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{n_e(x)-1}{2(n_e(x)+1)}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}$$

1.2.2) $\tilde{y}_k$ is the first point $y_{e,v}$ along the edge e. Notice that $r_{e,v}(x) = r_{e,w}(x) = 1$ while $r_{e,v}(x)$ and $r_{e,w}(y)$ go both to zero as $\tilde{y}$ tends to $\tilde{x}$. Moreover, $n_e(y) = n_e(x) + 2$. Hence,

$$\begin{aligned}
&\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i = \\
&\lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\Big(1 - \frac{n_e(y)-2}{n_e(y)-1}\Big)\Big(-\frac{1}{2} + \frac{r_{e,v}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\
&+ \frac{n_e(y)-2}{n_e(y)-1}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big)\Big) \\
&= \alpha_{e,v,w}\Big(-\frac{1}{2(n_e(y)-1)} + \frac{n_e(y)-2}{2(n_e(y)-1)}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{n_e(y)-3}{2(n_e(y)-1)}\Big) = \alpha_{e,v,w}\Big(\frac{n_e(x)-1}{2(n_e(x)+1)}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}$$

If $v \notin x$, let k be such that $\tilde{x}_k$ is the first point $x_{e,v}$ of x along e and then, $\tilde{y}_k$ is the first point $y_{e,v}$ of y along e and $n_e(y) = n_e(x) + 1$. Notice that $r_{e,w}(x) = 1$ and $r_{e,w}(y)$ goes to zero as $\tilde{y}$

tends to $\tilde{x}$. Hence,

$$\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i =$$
$$\lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\Big(1-\frac{n_e(y)-2}{n_e(y)-1}\Big)\Big(-\frac{1}{2}+\frac{r_{e,v}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big)$$
$$+\frac{n_e(y)-2}{n_e(y)-1}\Big(\frac{1}{2}-\frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big)\Big)$$
$$=\alpha_{e,v,w}\Big(\frac{1}{n_e(x)}\Big(-\frac{1}{2}+\frac{r_{e,v}(x)}{r_{e,v}(x)+n_e(x)}\Big)+\frac{n_e(x)-1}{2n_e(x)}\Big)$$
$$=\alpha_{e,v,w}\Big(\frac{n_e(x)+r_{e,v}(x)-2}{2(r_{e,v}(x)+n_e(x)}\Big)$$
$$=\alpha_{e,v,w}\Big(\frac{1}{2}-\frac{r_{e,w}(x)}{r_{e,v}(x)+r_{e,w}(x)+n_e(x)-1}\Big)=\widetilde{\Phi}(\tilde{x})_i.$$

1.3) Consider the case when $\tilde{y}_j$ is the last point $y_{e',w}$ of y along an edge e' which has w as an endpoint and such that $\delta_{e',w}(y) < \delta_{e,w}(y)$. Then, $\tilde{y}_i = y_{e,w}$, $\tilde{y}_j = y_{e',w}$.
If $v \in x$, then $v = \tilde{x}_k$ for some k and in this case we have the following possibilities for $\tilde{y}_k$.

1.3.1) $\tilde{y}_k$ is the vertex v or the first point $y_{e',v}$ for some edge e' such that $\delta_{e',v} < \delta_{e,v}$. In this case, notice that $n_e(y) = n_e(x)$, $r_{e,v}(x) = r_{e,w}(x) = 1$ and also $r_{e,v}(y) = r_{e,w}(y) = 1$. Hence,

$$\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i = \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w}$$
$$=\lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2}-\frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big)$$
$$=\alpha_{e,v,w}\Big(\frac{1}{2}-\frac{1}{n_e(y)+1}\Big)=\alpha_{e,v,w}\Big(\frac{1}{2}-\frac{1}{n_e(x)+1}\Big)$$
$$=\widetilde{\Phi}(\tilde{x})_i.$$

1.3.2) $\tilde{y}_k$ is the first point $y_{e,v}$ along the edge e. Notice that $r_{e,v}(x) = r_{e,w}(x) = 1$ and $r_{e,w}(y) = 1$, while $r_{e,v}(x)$ goes to zero as $\tilde{y}$ tends to $\tilde{x}$. Moreover, $n_e(y) = n_e(x) + 1$. Hence,

$$
\begin{aligned}
\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{r_{e,v}(y)+n_e(y)}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{n_e(y)}\Big) = \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{n_e(x)+1}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}
$$

If $v \notin x$, then let k be such that $\tilde{x}_k$ is the first point $x_{e,v}$ of x along e and then, $\tilde{y}_k$ is the first point $y_{e,v}$ of y along e. Notice that $n_e(y) = n_e(x)$, $r_{e,w}(x) = r_{e,w}(y) = 1$ and $r_{e,v}(y)$ goes to $r_{e,v}(x)$ as $\tilde{y}$ tends to $\tilde{x}$. Then,

$$
\begin{aligned}
\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{r_{e,v}(y)+n_e(y)}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{1}{r_{e,v}(x)+n_e(x)}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}
$$

2) if $w \notin x$, then also $w \notin y$.
If $v \in x$, then $v = \tilde{x}_j$ for some j and in this case we have the following possibilities for $\tilde{y}_j$.

2.1) $\tilde{y}_j$ is the vertex v or the first point $y_{e',v}$ for some edge e' such that $\delta_{e',v} < \delta_{e,v}$. Notice that $n_e(y) = n_e(x)$, $r_{e,v}(x) = r_{e,v}(y) = 1$ and $r_{e,w}(y)$ goes to $r_{e,w}(x)$ as $\tilde{y}$ tends to $\tilde{x}$. Then,

$$
\begin{aligned}
\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i &= \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y)+r_{e,w}(y)+n_e(y)-1}\Big) \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,w}(y)+n_e(y)}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(x)}{r_{e,w}(x)+n_e(x)}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}
$$

2.2) $\tilde{y}_j$ is the first point $y_{e,v}$ along the edge e, then $n_e(y) = n_e(x) + 1$, $r_{e,v}(x) = 1$ while $r_{e,v}(y)$ goes to zero as $\tilde{y}$ tends to $\tilde{x}$. Then,

$$\begin{aligned}
&\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i = \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y) + r_{e,w}(y) + n_e(y) - 1}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(x)}{r_{e,w}(x) + n_e(y) - 1}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(x)}{r_{e,w}(x) + n_e(x)}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}$$

If $v \notin x$, then let j be such that $\tilde{x}_j$ is the first point $x_{e,v}$ of x along e and then, $\tilde{y}_j$ is the first point $y_{e,v}$ of y along e and $n_e(y) = n_e(x)$. Hence,

$$\begin{aligned}
&\lim_{\tilde{y}\to\tilde{x}} \widetilde{\Phi}(\tilde{y})_i = \lim_{\tilde{y}\to\tilde{x}} \Phi(y)_{e,w} \\
&= \lim_{\tilde{y}\to\tilde{x}} \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(y)}{r_{e,v}(y) + r_{e,w}(y) + n_e(y) - 1}\Big) \\
&= \alpha_{e,v,w}\Big(\frac{1}{2} - \frac{r_{e,w}(x)}{r_{e,v}(x) + r_{e,w}(x) + n_e(x) - 1}\Big) = \widetilde{\Phi}(\tilde{x})_i.
\end{aligned}$$

□

Proposition 3.2.3. *Let G be a connected graph and Φ defined on $\mathcal{C}_n(G)$, then* $\operatorname{Im}\Phi = \mathcal{N}_n(G)$.

Proof. ($\subseteq$) Let $y \in \mathcal{C}_n(G)$ be a configuration in $\operatorname{Im}\Phi$. Then, $y \in \mathcal{N}_n(G)$ since all the conditions of Prop. 3.1.4 are satisfied by the definition of function Φ.

($\supseteq$) Let $y \in \mathcal{N}_n(G)$, then, for each edge $e \in E$, $-\frac{1}{2} < t_1 \leq \bar{t}_{e,v}$ and $\bar{t}_{e,w} \leq t_{n_e(x)} < \frac{1}{2}$. Notice that the values of $\Phi(x)_{e,v}$ and $\Phi(x)_{e,w}$ can be computed for each edge e of G, so by continuity of function Φ we can always choose another configuration $x \in \mathcal{C}_n(G)$ such that $x_V = y_V$, $n_e(x) = n_e(y)$, $y_1 = \Phi(x)_{e,v}$ and $y_{n_e(y)} = \Phi(x)_{e,w}$ for each $e \in E$. Hence, $\Phi(x) = y$, so $y \in \operatorname{Im}\Phi$.
□

Proposition 3.2.4. Φ *gives a weak deformation of $\mathcal{C}_n(G)$ into $\mathcal{N}_n(G)$.*

Proof. Remind that any configuration x can be written as $x = x_V \sqcup_{e \in E} x_e$, so we rewrite $\Phi(x) = \Phi(x)_V \sqcup_{e\in E} \Phi(x)_e$.

Let us define a homotopy $h_t : \mathcal{C}_n(G) \to \mathcal{C}_n(G)$ such that

$$h_t(x) = (1-t)x + t\Phi(x) \tag{3.2.7}$$

for all $t \in [0,1]$, in the sense we are going to explain.

First, $\Phi(x)_V = x_V$ and hence $h_t(x)_V = x_V$ for all $t \in [0,1]$.
Assume $n_e(x) = k$, then also $n_e(\Phi(x)) = k$ and consider

$$x_e = \{x_1 = \alpha_{e,v,w}(r_1), \dots, x_k = \alpha_{e,v,w}(r_k)\}$$

with $-\frac{1}{2} < r_1 < \cdots < r_k < \frac{1}{2}$ and

$$\Phi(x)_e = \{y_1 = \alpha_{e,v,w}(s_1), \dots, y_k = \alpha_{e,v,w}(s_k)\}$$

with $-\frac{1}{2} < s_1 < \cdots < s_k < \frac{1}{2}$.
Let us set

$$z_i(t) = (1-t)x_i + ty_i = \alpha_{e,v,w}((1-t)r_i + ts_i)$$

for all $i = 1, \dots, k$ and for all $t \in [0,1]$.

Then,

$$h_t(x)_e = (1-t)(x_e) + t(\Phi(x)_e) = \{z_1(t), \dots, z_k(t)\}$$

for all $t \in [0,1]$.

Observe that

$$h_0(x)_e = \{z_i(0)\}_i = x_e$$

and also

$$h_1(x)_e = \{z_i(1)\}_i = \Phi(x)_e.$$

So far we have verified that $h_0 = id_{\mathcal{C}_n(G)}$ and $h_1(x) \in \operatorname{Im}\Phi$ for all $x \in \mathcal{C}_n(G)$. Now we also verify that $h_t(x) \in \operatorname{Im}\Phi$ for all $x \in \operatorname{Im}\Phi$ and for all $t \in [0,1]$.
Let us consider a configuration $x' \in \operatorname{Im}\Phi$ and let $x \in \mathcal{C}_n(G)$ be such that $x' = \Phi(x)$.
Again assume $n_e(x) = k$.
We consider

$$\Phi(x') = \Phi(x')_V \sqcup_{e\in E} \Phi(x')_e$$

and we set

$$x_e = \{x_1 = \alpha_{e,v,w}(r_1), \ldots, x_k = \alpha_{e,v,w}(r_k)\}$$

such that $r_1 < \cdots < r_k$,

$$x'_e = \{x'_1 = \alpha_{e,v,w}(s_1), \ldots, x'_k = \alpha_{e,v,w}(s_k)\}$$

such that $-\frac{1}{2} < s_1 < \cdots < s_k < \frac{1}{2}$ and

$$\Phi(x')_e = \{\Phi(x')_1 = \alpha_{e,v,w}(u_1), \ldots, \Phi(x')_k = \alpha_{e,v,w}(u_k)\}$$

such that $-\frac{1}{2} < u_1 < \ldots u_k < \frac{1}{2}$.

Observe that $h_t(x')_V = x'_V$ for all $t \in [0,1]$ since $\Phi(x')_V = x'_V$.

Then,

$$h_t(x')_{e,v} = \alpha_{e,v,w}((1-t)r_1 + tu_1)$$
$$h_t(x')_{e,w} = \alpha_{e,v,w}((1-t)r_k + tu_k)$$

for all $t \in [0,1]$. Hence, the coordinate of $h_t(x')_{e,v}$ is contained in the interval $(-\frac{1}{2}, \bar{r}_{e,v}]$ and the coordinate of $h_t(x')_{e,w}$ in $[\bar{r}_{e,w}, \frac{1}{2})$. Indeed, both r_1 and u_1 are contained in these intervals since x' and $\Phi(x')$ are contained in $\operatorname{Im}\Phi$. Thus, a convex combination of their coordinates is still contained in it.
Moreover, the coordinate of $h_t(x')_{e,v}$ coincides with $\bar{r}_{e,v}$ if there is some $h_t(x')_{e',v}$ such that $\delta_{e',v} < \delta_{e,v}$ for some edge $e' \neq e$. Indeed, condition *ii*) in Def. 3.1.4 holds for r_1 and u_1 since x' and $\Phi(x')$ are contained in $\operatorname{Im}\Phi$ and again a convex combination of their coordinates still satisfy the condition.
Analogous reasonings are valid with respect to the vertex w.

Then, notice that the intermediate points $x'_2, \ldots, x'_{k-1}$ are uniformly distributed between x'_1 and x'_k, so we can write their coordinates as

$$\left(1 - \frac{i}{k-1}\right) s_1 + \frac{i}{k-1} s_k$$

for $i = 1, \ldots, k-2$.

Analogously, the intermediate points $\Phi(x')_2, \ldots, \Phi(x')_{k-1}$ are uniformly distributed between $\Phi(x')_1$ and $\Phi(x')_k$, thus we can write their coordinates as

$$\left(1 - \frac{i}{k-1}\right) u_1 + \frac{i}{k-1} u_k$$

for $i = 1, \dots, k-2$.

Then, the intermediate points $h_t(x')_i = (1-t)x'_i + t\Phi(x')_i$ for $i = 1, \cdots, k-2$ are uniformly distributed since we can rewrite their coordinates as follows:

$$(1-t)\left(\left(1-\frac{i}{k-1}\right)s_1 + \frac{i}{k-1}s_k\right) + t\left(\left(1-\frac{i}{k-1}\right)u_1 + \frac{i}{k-1}u_k\right)$$
$$= \left(1-\frac{i}{k-1}\right)((1-t)s_1 + tu_1) + \frac{i}{k-1}((1-t)s_k + tu_k)$$

for all $t \in [0,1]$.

Hence, $\operatorname{Im}\Phi$ is a weak deformation of $\mathcal{C}_n(G)$ and so $\mathcal{N}_n(G)$ is a weak deformation of $\mathcal{C}_n(G)$ since $\operatorname{Im}\Phi = \mathcal{N}_n(G)$. □

Given a connected graph G, consider the subset

$$V_1 = \{v \in V \mid \deg(v) \leq 1\}.$$

We denote by $\mathcal{C}'_n(G)$ the subset of $\mathcal{C}_n(G)$ consisting of the configurations of $\mathcal{C}_n(G)$ which satisfy for each vertex $v \in V_1$ and each edge $e \in E$ with $v \in e$,

$$x_e = \{x_1 = \alpha_{e,v,w}(t_1), \dots, x_{n_e(x)} = \alpha_{e,v,w}(t_{n_e(x)})\}$$

with $\bar{t}_{e,v} \leq t_1 < \cdots < t_{n_e(x)} < \frac{1}{2}$.

Proposition 3.2.5. $\Phi(\mathcal{C}'_n(G))$ *is a weak deformation of* $\mathcal{C}'_n(G)$.

Proof. The proof is analogous to that of Prop. 3.2.4 except for the edges containing a terminal vertex. Let $x \in \mathcal{C}'_n(G)$ and consider an edge e containing $v \in V_1$. Then,

$$x_e = \{x_1 = \alpha_{e,v,w}(t_1), \dots, x_{n_e(x)} = \alpha_{e,v,w}(t_{n_e(x)})\}$$

with $\bar{t}_{e,v} \leq t_1 < \cdots < t_{n_e(x)} < \frac{1}{2}$. Then it is sufficient to observe that also the image

$$\Phi(x)_e = \left\{y_1 = \alpha_{e,v,w}(u_1), \dots, y_k = \alpha_{e,v,w}(u_{n_e(x)})\right\}$$

with $\bar{u}_{e,v} \leq u_1 < \cdots < u_{n_e(x)} < \frac{1}{2}$ is contained in $\mathcal{C}'_n(G)$. Thus, the proposition follows again using the same reasoning of Prop. 3.2.4. □

3.3 The cubical complex $\mathcal{Q}_n(G)$

Proposition 3.3.1. *Given a connected graph G, we can associate to the normalized configuration space $\mathcal{N}_n(G)$ a cubical complex structure which we denote by $\mathcal{Q}_n(G)$. In particular,* $\dim \mathcal{Q}_n(G) \leq \min\{n, |V|\}$.

Proof. In order to construct $\mathcal{Q}_n(G)$ we describe its cells.
The 0-cells of $\mathcal{Q}_n(G)$ are the configurations in $\mathcal{C}_n(G)$ satisfying the conditions below.
For each edge e whose vertices are v and w, consider the parametrization $\alpha_{e,v,w}$ of e from v to w and $x_e = \{x_1 = \alpha_{e,v,w}(r_1), \ldots, x_k = \alpha_{e,v,w}(r_k)\}$ such that $-\frac{1}{2} < r_1 < \cdots < r_k < \frac{1}{2}$ assuming $n_e(x) = k$. Then,

i) $r_1 = \bar{r}_{e,v}$ where $\bar{r}_{e,v}$ is defined as $\bar{t}_{e,v}$ in (3.1.2);

ii) $r_k = \bar{r}_{e,w}$ where $\bar{r}_{e,w}$ is defined as $\bar{t}_{e,w}$ in (3.1.2);

iii) the intermediate points are uniformly distributed along e between x_1 and x_k.

Notice that any configuration x satisfying the conditions above can be uniquely determined by $\nu(x)$, hence, we can identify each 0-cell with the couple $((n_v(x))_{v\in V}, (n_e(x))_{e\in E})$ such that $\sum_v n_v + \sum_e n_e = n$.
Let $x = \big(n_v(x), n_e(x)\big)$ and $x' = \big(n_v(x'), n_e(x')\big)$ be two 0-cells of $\mathcal{Q}_n(G)$ which, for an edge e_0 and a vertex w_0 such that $w_0 \in e_0$, satisfy the following conditions:

$$n_v(x') = \begin{cases} n_v(x) + 1 & \text{if } v = w_0 \\ n_v(x) & \text{otherwise} \end{cases}$$

$$n_e(x') = \begin{cases} n_e(x) - 1 & \text{if } e = e_0 \\ n_e(x) & \text{otherwise.} \end{cases}$$

Then a 1-cell s of $\mathcal{Q}_n(G)$ is an oriented edge from x to x'.

Let v_0 and w_0 be the vertices of e_0 and $\alpha_{e_0,v_0,w_0}\colon [-\frac{1}{2}, \frac{1}{2}] \to e_0$ be the parametrization of e_0 such that $\alpha_{e_0,v_0,w_0}(-\frac{1}{2}) = v_0$ and $\alpha_{e_0,v_0,w_0}(\frac{1}{2}) = w_0$. Assume $n_{e_0}(x) = k$ and hence $n_{e_0}(x') = k - 1$.
Consider

$$x_{e_0} = \{x_1 = \alpha_{e_0,v_0,w_0}(r_1), \ldots, x_k = \alpha_{e_0,v_0,w_0}(r_k)\}$$

and

$$x'_{e_0} = \{x'_1 = \alpha_{e_0,v_0,w_0}(r'_1), \ldots, x'_{k-1} = \alpha_{e_0,v_0,w_0}(r'_{k-1})\}.$$

We set

$$x'_k = \alpha_{e_0,v_0,w_0}(r'_k) = \alpha_{e_0,v_0,w_0}(\frac{1}{2}) = w_0.$$

Consider the injective parametrization $\beta_s \colon [0,1] \to \mathcal{C}_n(G)$ such that $\beta_s(0) = x$, and $\beta_s(1) = x'$ defined by

$$\beta_s(t)_i = \alpha_{e_0,v_0,w_0}\left((1-t)\, r_i + t r'_i\right)$$

for $i = 1, \ldots, k$.

Remark 3.3.2. If e_0 is not a loop, there is only one possibility to choose the parametrization α_{e_0,v_0,w_0} in order have $n_{w_0}(x) = 0$ and $n_{w_0}(x') = 1$. Hence, there is exactly one 1-cell from x to x'.

If e_0 is a loop with a single vertex w_0, then there are two opposite parametrizations of e_0 and hence two different points on e_0 can be moved towards the vertex w_0. Hence, we have two distinct 1-cells of $\mathcal{Q}_n(G)$ from x to x'.

A 2-cell a of the cubical complex $\mathcal{Q}_n(G)$ is a square attached to four 0-cells $x = \big(n_v(x), n_e(x)\big)$, $y = \big(n_v(y), n_e(y)\big)$, $y' = (n_v(y'), n_e(y')), z = (n_v(z), n_e(z))$ and four 1-cells s_1, s'_1, s_2, s'_2 of $\mathcal{Q}_n(G)$ which satisfy the following conditions.
Let us consider two distinct edges $e_0, e'_0 \in E$ and two distinct vertices $w_0, w'_0 \in V$. Then, the four 0-cells must satisfy:

$$n_v(y) = \begin{cases} n_v(x) + 1 & \text{if } v = w_0 \\ n_v(x) & \text{otherwise,} \end{cases}$$

$$n_e(y) = \begin{cases} n_e(x) - 1 & \text{if } e = e_0 \\ n_e(x) & \text{otherwise,} \end{cases}$$

and

$$n_v(y') = \begin{cases} n_v(x) + 1 & \text{if } v = w'_0 \\ n_v(x) & \text{otherwise,} \end{cases}$$

$$n_e(y') = \begin{cases} n_e(x) - 1 & \text{if } e = e'_0 \\ n_{e'}(x) & \text{otherwise,} \end{cases}$$

$$n_v(z) = \begin{cases} n_v(x)+1 & \text{if } v = w_0, w_0' \\ n_v(x) & \text{otherwise,} \end{cases}$$

$$n_e(z) = \begin{cases} n_e(x)-1 & \text{if } e = e_0, e_0' \\ n_e(x) & \text{otherwise.} \end{cases}$$

The four 1-cells are the oriented edges s_1 from x to y, s_1' from x to y', s_2 from y to z and s_2' from y' to z.

Take two parametrizations α_{e_0,v_0,w_0} and $\alpha_{e_0',v_0',w_0'}$ respectively of the edges e_0 and e_0' and consider $x_{e_0} = \{x_1, \dots, x_k\}$ and $x_{e_0'} = \{x_1', \dots, x_{k'}'\}$. For the edges s_1, s_1', s_2, s_2' we consider the parametrizations

$$\beta_{s_1} : t_1 \to \mathcal{C}_n(G) \quad \text{and} \quad \beta_{s_2'} : t_1 \to \mathcal{C}_n(G)$$

depending on the parameter t_1 and

$$\beta_{s_2} : t_2 \to \mathcal{C}_n(G) \quad \text{and} \quad \beta_{s_1'} : t_2 \to \mathcal{C}_n(G)$$

depending on the parameter t_2 as defined before. Notice that the points of x_{e_0} moves indipendently from those of $x_{e_0'}$since $e_0 \neq e_0'$. Then, we define $\beta_a \colon [0,1]^2 \to \mathcal{C}_n(G)$ sending $t = (t_1, t_2) \in [0,1]^2$ to the unique configuration $\beta_a(t)$ such that

$$\begin{aligned} \beta_a(t) \cap \bar{e}_0 &= \beta_{s_1}(t_1) \cap \bar{e}_0 = \beta_{s_2'}(t_1) \cap \bar{e}_0 \\ \beta_a(t) \cap \bar{e}_0' &= \beta_{s_1'}(t_2) \cap \bar{e}_0' = \beta_{s_2}(t_2) \cap \bar{e}_0' \\ \beta_a(t) \cap (G - \bar{e}_0 \cup \bar{e}_0') &= x \cap (G - \bar{e}_0 \cup \bar{e}_0'). \end{aligned}$$

It is also possible to define β_a explicitely using the following notation which include directly the dependence on the parameters:

$$\beta_a(t_1, t_2) = x^{t_1 t_2}.$$

In this way we denote the 0-cells by $x^{00} = x$, $x^{10} = y$, $x^{01} = y'$ and $x^{11} = z$. Then, we can write

$$x_i^{t_1 t_2} = (1-t_1)(1-t_2)x_i^{00} + t_1(1-t_2)x_i^{10} + (1-t_1)t_2 x_i^{01} + t_1 t_2 x_i^{11} \quad (3.3.8)$$

for $i = 1, \dots, n_{e_0}(x)$.

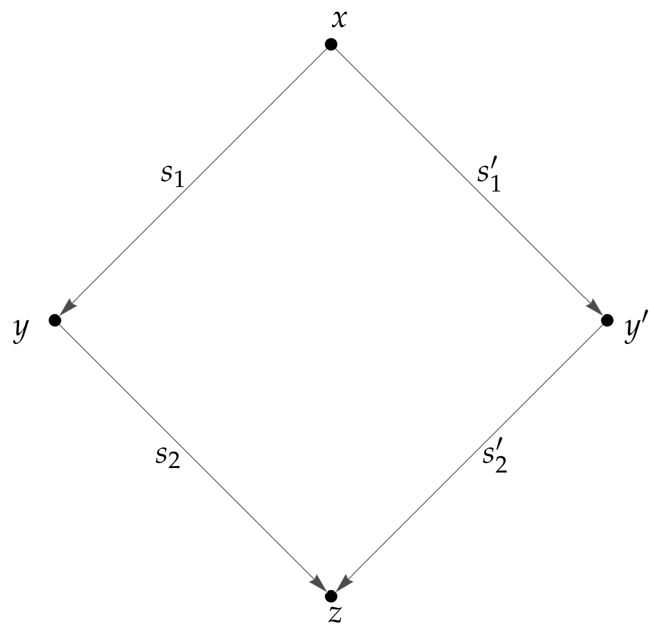

Figure 3.3.4: *A 2-cell of* $\mathcal{Q}_n(G)$.

So far we have assumed $e_0 \neq e'_0$, now consider $e_0 = e'_0$ and see how the construction above must be modified.
Let $e_0 \in E$ have the vertices v_0 and w_0 as endpoints. Then $w'_0 = v_0$ since $w_0 \neq w'_0$. Hence the conditions on the 0-cells x, y, y' are the same while the conditions on z must be rewritten as follows:

$$n_v(z) = \begin{cases} n_v(x) + 1 & \text{if } v = w_0, w'_0 \\ n_v(x) & \text{otherwise} \end{cases}$$

$$n_e(z) = \begin{cases} n_e(x) - 2 & \text{if } e = e_0 \\ n_e(x) & \text{otherwise.} \end{cases}$$

In this case, the points of x on e_0 move depending on both parameters t_1 and t_2 at the same time and we can use the same formula (3.3.8) to define $x^{t_1 t_2}$.

It is possible to generalize this reasoning to construct an m-cell c of $\mathcal{Q}_n(G)$. In this case we denote with $x^{i_1 \cdots i_m}$ the 0-cells of $\mathcal{Q}_n(G)$ with $i_1, \ldots, i_m = 0, 1$ depending on the value of parameters $t_1, \ldots, t_m$.
Then, as before if all the edges $e_1, \ldots, e_m$ are distinct, then the points on each edge e_i of configuration $x^{t_1, \ldots, t_m}$ depend only on one parameter t_i at a time, while if there are some coincident edges, we need a combination

of pairs of parameters as seen in the case of the 2-cells.

By the parametrizations defined above, we have constructed the cubical complex $\mathcal{Q}_n(G)$ directly inside the configuration space $\mathcal{C}_n(G)$ in such a way that coincides with $\mathcal{N}_n(G)$.

Notice that an m-cell of $\mathcal{Q}_n(G)$ indicates that there are m points on the interiors of some edges of G which are moving towards m vertices of G. Hence, m cannot exceed the number of points n on G or the number of vertices $|V|$ of G. □

Proposition 3.3.3. *Given the cubical complex $\mathcal{Q}_n(G)$, a presentation for the fundamental group of the 2-skeleton $\pi_1(\mathcal{Q}_n^2(G))$ is a presentation for the the n-braid group $\mathcal{B}_n(G)$.*

Proof. First we prove that $\mathcal{B}_n(G)$ is isomorphic to $\pi_1(\mathcal{Q}_n(G))$. By Prop. 3.3.1, $\mathcal{Q}_n(G)$ coincides with $\mathcal{N}_n(G)$ and by Prop. 3.2.4, $\mathcal{N}_n(G)$ is a weak deformation of $\mathcal{C}_n(G)$, hence,

$$\mathcal{B}_n(G) \cong \pi_1(\mathcal{C}_n(G)) \cong \pi_1(\mathcal{N}_n(G)) \cong \pi_1(\mathcal{Q}_n(G)).$$

Then, in order to get a presentation for the braid group $\mathcal{B}_n(G)$ we can compute a presentation for the fundamental group $\pi_1(\mathcal{Q}_n(G))$ and by Prop. 1.3.5 and Prop. 1.4.5 we can consider just the 2-skeleton of $\mathcal{Q}_n(G)$.

□

Definition 3.3.4. Let $\mathcal{Q}'_n(G)$ be the subcomplex of $\mathcal{Q}_n(G)$ obtained eliminating all the 0-cells of $\mathcal{Q}_n(G)$ of the form $x = (n_v(x), n_e(x))_{v\in V, e\in E}$ such that there is at least one terminal vertex $v_0 \in V_1$ with $n_{v_0}(x) = 1$ and eliminating all the m-cells containing those 0-cells, for $m \geq 1$. In particular, dim $\mathcal{Q}'_n(G) \leq \min(n, \ell)$ where $\ell = |V - V_1|$.

Notice that the dimension of $\mathcal{Q}'_n(G)$ is bounded in an analogous way with respect to $\mathcal{Q}_n(G)$.

Proposition 3.3.5. *Given the cubical subcomplex $\mathcal{Q}'_n(G)$, a presentation for the fundamental group of the 2-skeleton of $\mathcal{Q}'_n(G)$ is a presentation for the n-braid group $\mathcal{B}_n(G)$.*

Proof. The definition given for the cubical subcomplex $\mathcal{Q}'_n(G)$ is equivalent to construct the cubical complex which coincides with $\mathcal{C}'_n(G)$. By Prop. 3.2.5, $\mathcal{C}'_n(G)$ is a weak deformation of $\mathcal{C}_n(G)$, hence

$$\mathcal{B}_n(G) \cong \pi_1(\mathcal{C}_n(G)) \cong \pi_1(\mathcal{C}'_n(G)) \cong \pi_1(\mathcal{Q}'_n(G)).$$

Then, by Prop. 1.3.5 and Prop. 1.4.5 we are able to compute a presentation for $\pi_1(\mathcal{Q}'_n(G))$ considering the 2-skeleton of $\mathcal{Q}'_n(G)$.

□

Corollary 3.3.6. *For any connected graph G not homeomorphic to S^1 and such that G contains one single vertex v_0 of degree $k > 2$, then the braid group $\mathcal{B}_n(G)$ is a free group.*

Proof. By hypothesis $|V - V_1| = 1$, so $\dim \mathcal{Q}'_n(G) \leq \min(n, 1)$ which means that the subcomplex $\mathcal{Q}'_n(G)$ is a graph. Then, by Prop. 1.4.2 the braid group $\mathcal{B}_n(G)$ is a free group. □

Chapter 4

Applications and examples

In this chapter we analyze some classes of connected graphs and we use the cubical subcomplexes $\mathcal{Q}'_n(G)$ in order to obtain presentations for the braid groups on those graphs.
First, we discuss the family T_k of radial trees consisting of k edges and we prove again the results in Prop. 2.3.4 and Prop. 2.3.3 already seen in Chapter 2. Then, we find analogous results for the family L_k of bouquets of k loops.
For both these families we include some examples and some figures of the corresponding cubical subcomplexes $\mathcal{Q}'_n(T_k)$ and $\mathcal{Q}'_n(L_k)$ made using Wolfram Mathematica. After that, we compute some presentations for the braid groups of other examples and we compare them to already known outcomes whenever possible. In particular we see the simplest graph T which contradicts Ghrist's conjecture, [Gh99].
To do the computation, we use a Mathematica code which first calculates a presentation for the fundamental group of the cubical subcomplex $\mathcal{Q}'_n(G)$ as proved in Prop.1.4.5, and then, when needed, it applies the Tiezte transformations in order to simplify the presentation. The code used either for computing the presentations and for creating the figures can be found in detail in Appendix A.

4.1 Radial trees T_k

Let T_k be a radial connected graph consisting of $k \geq 3$ terminal vertices $v_1, \dots, v_k$, a vertex v_0 of degree k and k edges $e_1, \dots, e_k$ such that each e_i joins v_0 and v_i for $i = 1, \dots, k$.

Let us consider the cubical subcomplex $\mathcal{Q}'_n(G)$ and notice that by Corol. 3.3.6, it is a graph since it has a single vertek of degree greater than 2.

Proposition 4.1.1. *The braid group $\mathcal{B}_n(T_k)$ is isomorphic to a free group on*

$$1-\frac{(n+k-2)!(2n+k-nk-1)}{n!(k-1)!}$$

generators.
The pure braid group $\mathcal{P}_n(T_k)$ is isomorphic to a free group on

$$1-\frac{(n+k-2)!(2n+k-nk-1)}{(k-1)!}$$

generators.

Proof. By Prop. 3.3.5, $\mathcal{B}_n(T_k) \cong \pi_1(\mathcal{Q}'_n(T_k))$ and by Prop. 1.4.3 we need to compute the Euler characteristic of $\mathcal{Q}'_n(T_k)$ to find the number of generators of the free group.
First, we observe that the vertices of $\mathcal{Q}'_n(T_k)$ are the configurations satisfying one of the following characteristics:

i) one point lays on v_0 and the other $n-1$ points are distributed in the interiors of the k edges $e_1, \ldots, e_k$;

ii) all the n points are distributed in the interiors of the k edges $e_1, \ldots, e_k$.

Then, we count the vertices satisfying $i)$ as $(k-1)$-combinations with repetitions of $n-1$ elements and hence they are exactly $\binom{n+k-2}{n-1}$. Similarly we count the vertices satisfying $ii)$ as k-combinations with repetitions of n elements and hence they are exactly $\binom{n+k-1}{n}$. Hence, there are

$$\binom{n+k-2}{n-1}+\binom{n+k-1}{n}$$

0-cells of $\mathcal{Q}'_n(T_k)$.

The 1-cells of $\mathcal{Q}'_n(T_k)$ are the oriented edges from a vertex of type $ii)$ to a vertex of type $i)$.
Notice that each vertex of type $i)$ has degree equal to k, indeed the point which occupies v_0 can come from any of the k edges of T_k.
In a similar way, each vertex of type $ii)$ has degree equal to the number of non-empty edges of T_k, indeed each non-empty edge of T_k has one point which can occupy the vertex v_0.
Then, the total number of 1-cells of $\mathcal{Q}'_n(T_K)$ is:

$$\frac{1}{2}\left(k\binom{n+k-2}{n-1}+\sum_{l=1}^{k} l\binom{k}{l}\binom{n-1}{n-l}\right)=k\binom{n+k-2}{n-1}.$$

Finally, the Euler characteristic of $\mathcal{Q}'_n(T_k)$ is

$$\begin{aligned}\chi(\mathcal{Q}'_n(T_k)) =& \binom{n+k-2}{n-1} + \binom{n+k-1}{n} - k\binom{n+k-2}{n-1} \\ =&(1-k)\frac{(n+k-2)!}{(n-1)!(k-1)!} + \frac{(n+k-1)!}{n!(k-1)!} \\ =&\frac{(n+k-2)!(2n+k-nk-1)}{n!(k-1)!}\end{aligned}$$

Hence, the braid group $\mathcal{B}_n(T_k)$ is isomorphic to a free group on

$$1-\chi(\mathcal{Q}'_n(T_k)) = 1 - \frac{(n+k-2)!(2n+k-nk-1)}{n!(k-1)!} \tag{4.1.9}$$

generators. Then, notice that if we multiply by a factor $n!$ the Euler characteristic for $\mathcal{Q}'_n(T_k)$, we obtain the Euler characteristic for the labeled configuration space $\widetilde{\mathcal{Q}}'_n(T_k)$. Hence, the number of generators for $\mathcal{P}_n(T_k)$ is

$$1-\chi(\widetilde{\mathcal{Q}}'_n(T_k)) = 1 - \frac{(n+k-2)!(2n+k-nk-1)}{(k-1)!}$$ □

These results agree with those already seen in Prop. 2.3.4 and in Prop. 2.3.3.

In particular, by (4.1.9), the braid group $\mathcal{B}_n(T_3)$ is isomorphic to a free group on $\frac{n(n-1)}{2}$ generators and the braid group $\mathcal{B}_n(T_4)$ is isomorphic to a free product on $\frac{n(n-1)(2n+5)}{6}$ generators.

Let us see some examples.
Consider the graph T_3 and rename it Y.
For $n=1$, the braid group $\mathcal{B}_1(Y) \cong 0$ since $\mathcal{Q}'_1(Y)$ is a tree isomorphic to Y.
For $n=2$, the cubical subcomplex $\mathcal{Q}'_2(Y)$ has three vertices of type $i)$ of degree 3 and six vertices of type $ii)$ arranged as in Figure 4.1.7.

Observe that there is only one generator since there is only one edge out of the chosen maximal spanning tree. Hence, the braid group $\mathcal{B}_2(Y)$ is isomorphic to a free group on one generator,

$$\mathcal{B}_2 Y \cong \pi_1(\mathcal{D}_2(Y)) \cong \mathbb{Z}.$$

For $n=3$, the cubical subcomplex $\mathcal{Q}'_3(Y)$ has six vertices of type $i)$ and ten vertices of type $ii)$ as in Figure 4.1.8. As soon as we choose a spanning

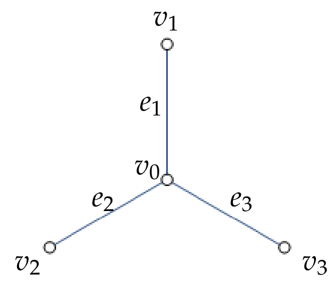

Figure 4.1.5: *Graph Y.*

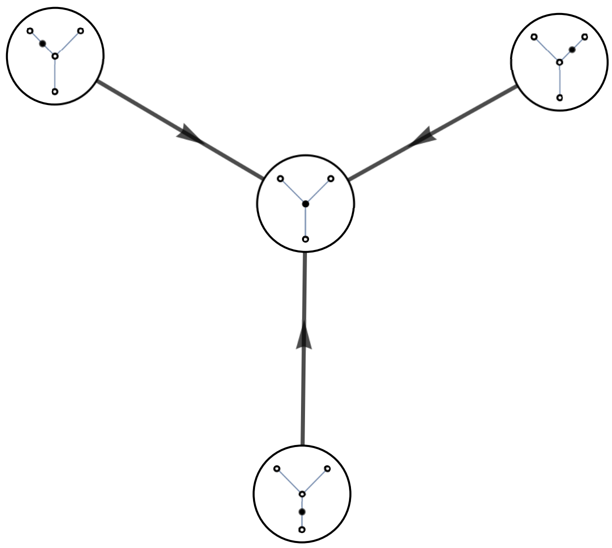

Figure 4.1.6: *The subcomplex $\mathcal{Q}'_1(Y)$.*

tree of $\mathcal{Q}'_3(Y)$ we notice that there are three edges out of it and hence a presentation for $\pi_1(\mathcal{Q}'_3(Y))$ has three generators. Thus, the braid group $\mathcal{B}_3(Y)$ is isomorphic to a free group on three generators,

$$\mathcal{B}_3 Y \cong \pi_1(\mathcal{D}_3(Y)) \cong \mathbb{F}_3.$$

For $n = 4$, observe in Figure 4.1.9 the subcomplex $\mathcal{Q}'_4(Y)$. Notice that there are six edges outside of the maximal tree. Hence, we get:

$$\mathcal{B}_4 Y \cong \mathbb{F}_6.$$

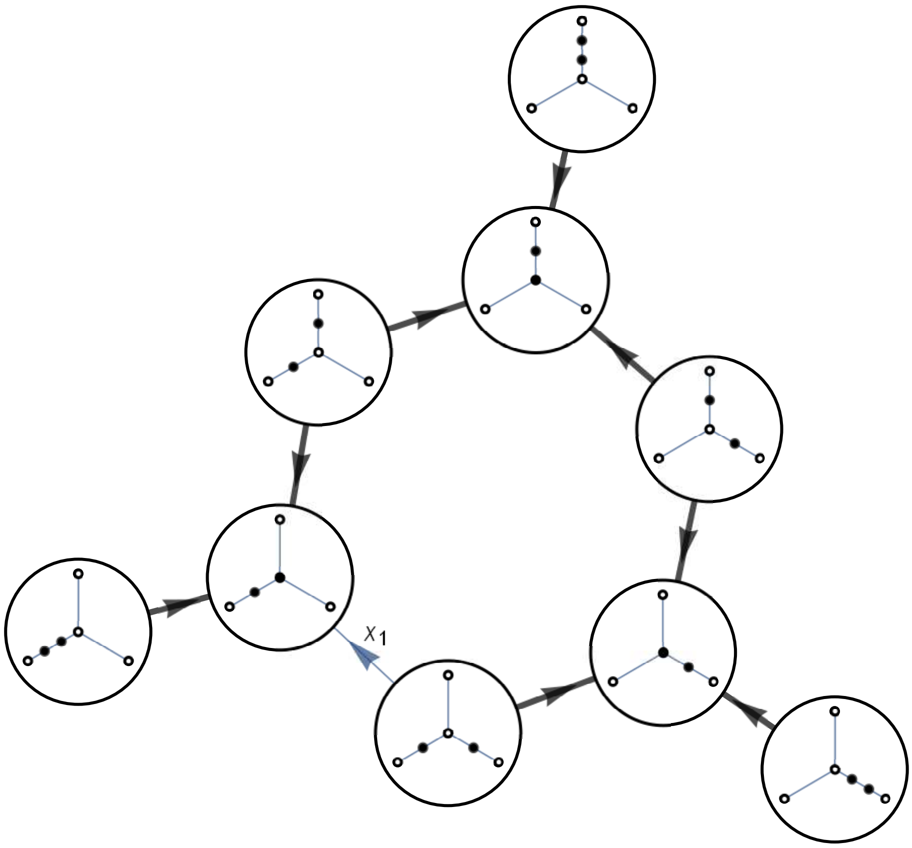

Figure 4.1.7: *The subcomplex $\mathcal{Q}_2'(Y)$ with the chosen maximal spanning tree in black.*

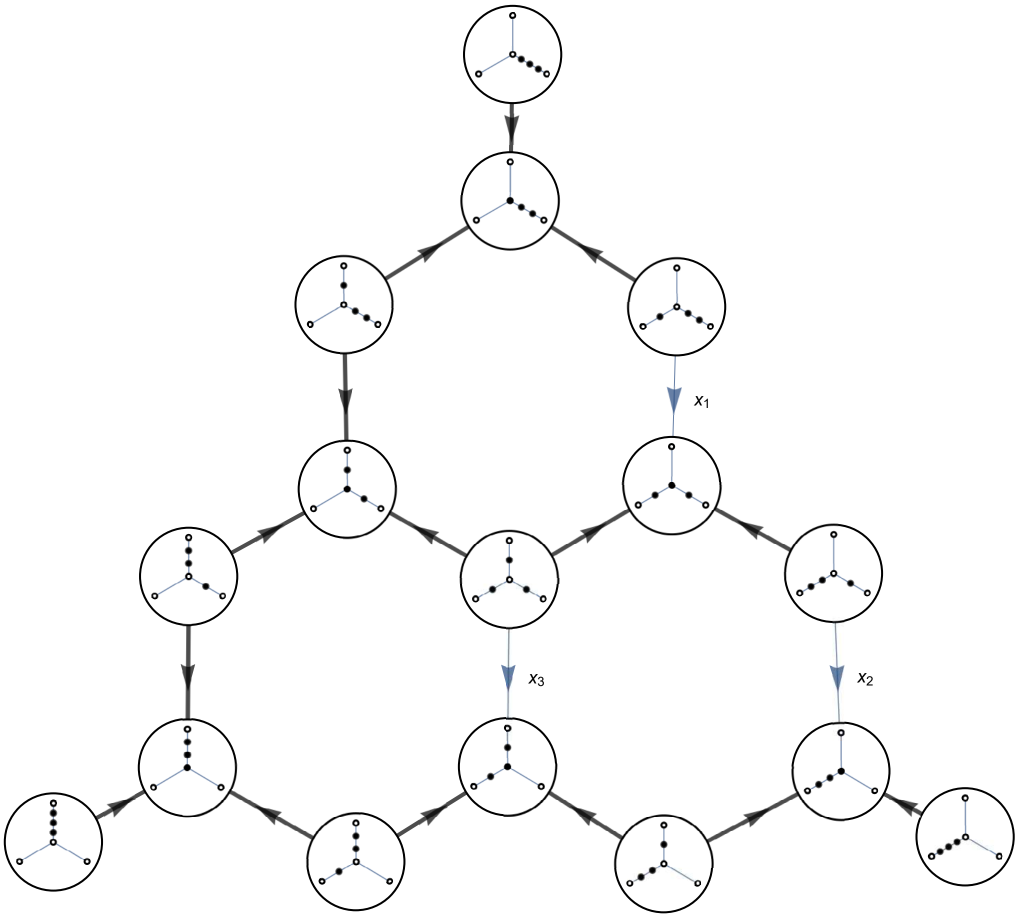

Figure 4.1.8: $\mathcal{Q}'_3(Y)$ *with the chosen maximal spanning tree in black.*

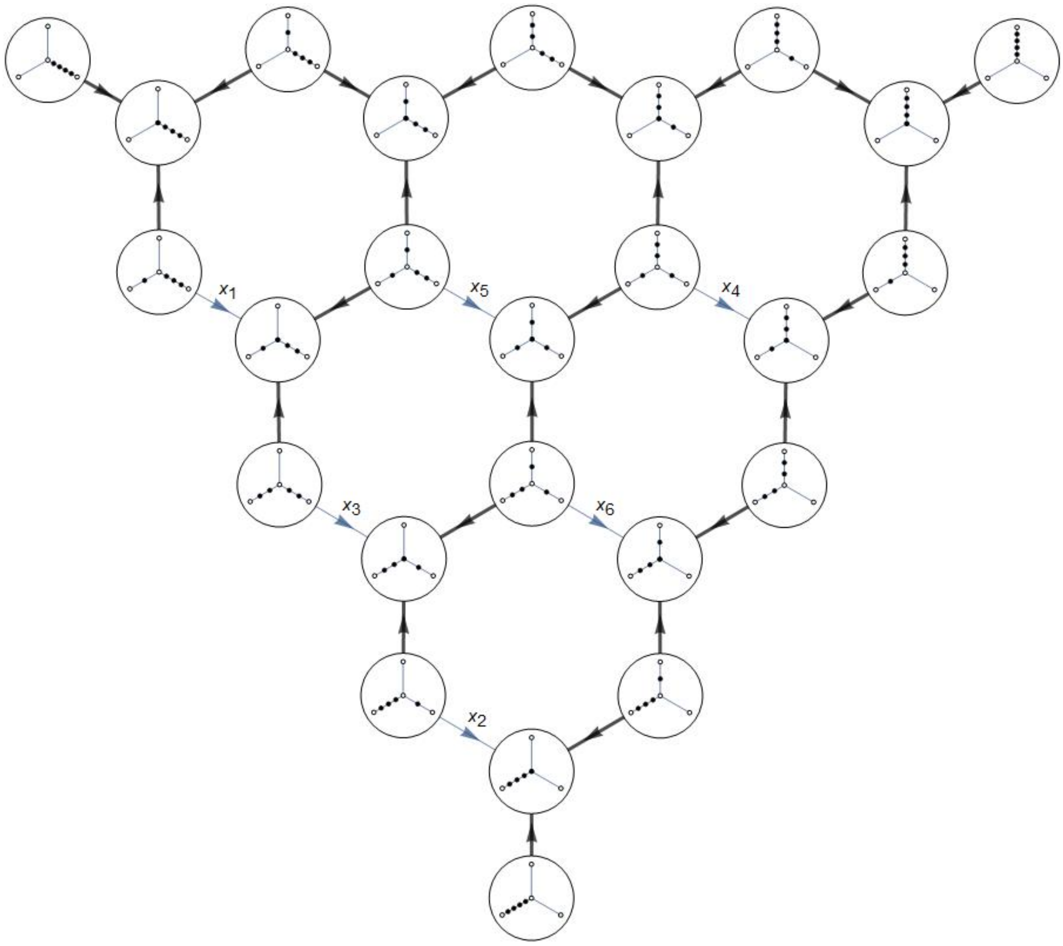

Figure 4.1.9: *The cubical complex $\mathcal{Q}_4'(Y)$ with the maximal tree chosen in black.*

Let us call X the graph T_4.

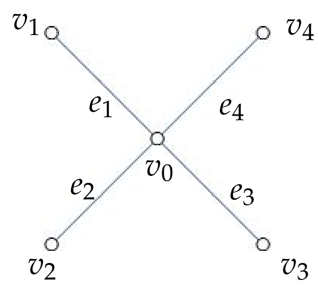

Figure 4.1.10: *Graph X.*

For $n = 1$, the braid group $\mathcal{B}_1(X)$ is trivial since $\mathcal{Q}'_n(X)$ is a tree isomorphic to X.

For $n = 2$, let us observe in Figure 4.1.11 the subcomplex $\mathcal{Q}'_2(X)$. Notice that there are three edges outside of the maximal tree. Hence, we have:

$$\pi_1(\mathcal{Q}'_2(X)) = \langle x_1, x_2, x_3 \rangle \text{ that is } \mathcal{B}_2(X) \cong \mathbb{F}_3.$$

In a similar way, if we consider $\mathcal{Q}'_3(X)$ then its fundamental group has a presentation consisting of eleven generators. Hence, $\mathcal{B}_3(X) \cong \mathbb{F}_{11}$.

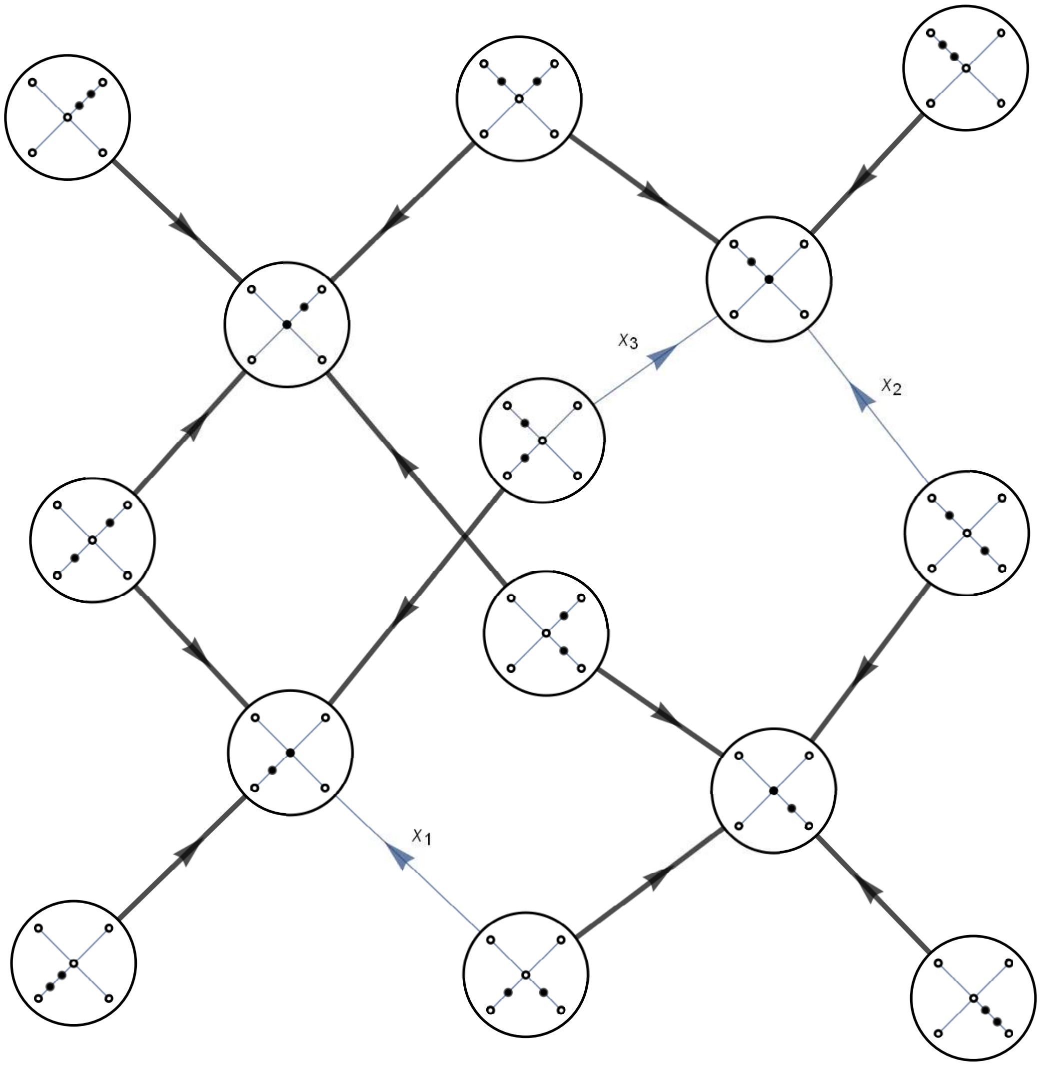

Figure 4.1.11: *The cubical subcomplex $\mathcal{Q}_2'(X)$ with the chosen maximal tree highlighted in black.*

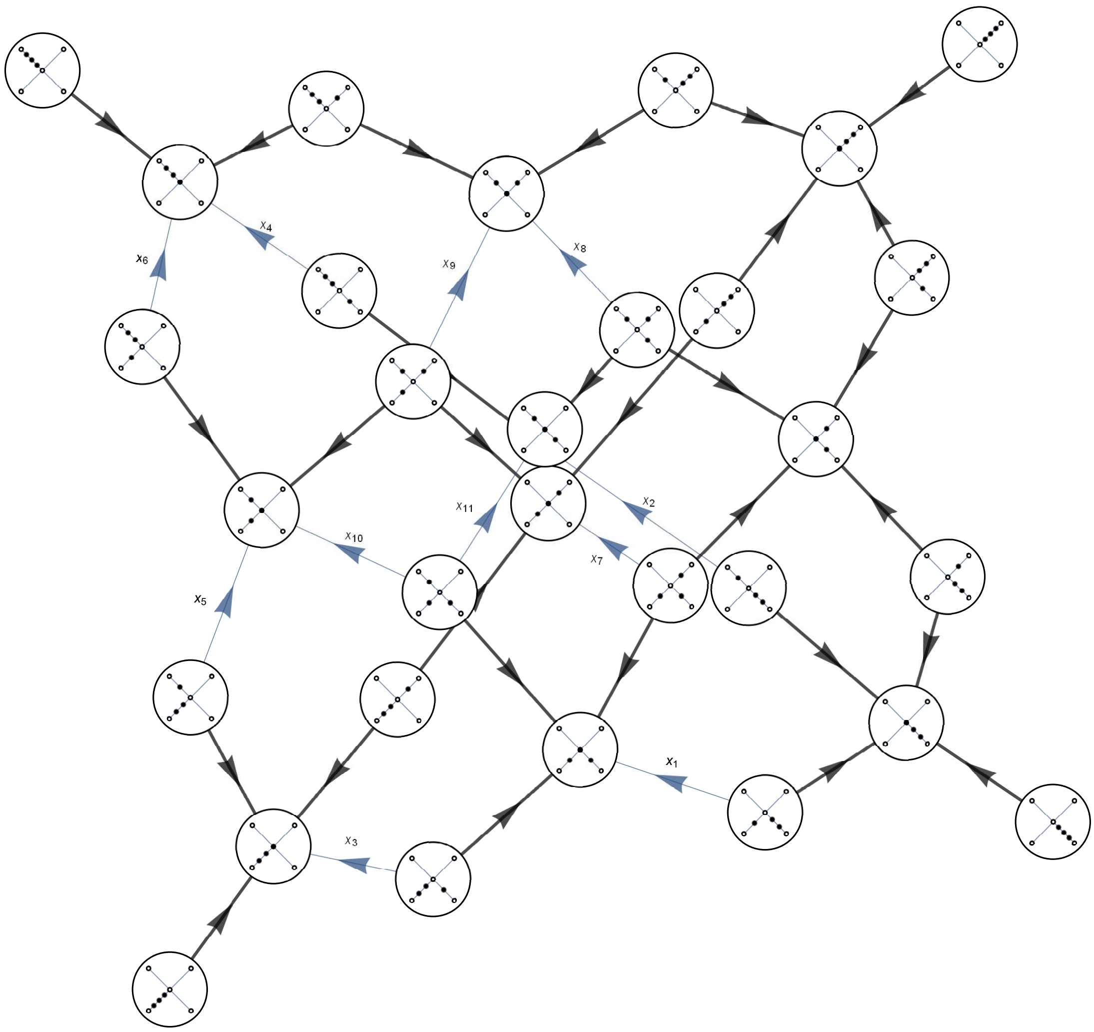

Figure 4.1.12: *The cubical subcomplex $\mathcal{Q}'_3(X)$ with the chosen maximal spanning tree in black.*

4.2 Bouquets of loops L_k

Let L_k be a graph consisting of k loops and a single vertex v_0 of degree $2k$. First notice that $\mathcal{Q}'_n(L_k) = \mathcal{Q}_n(L_k)$ since there are no terminal vertices in L_k and by Prop. 2.3.1, the dimension of $\mathcal{Q}_n(L_k)$ is equal to 1 since there is one single vertex of degree greater than 2, that is $\mathcal{Q}_n(L_k)$ is a graph.

Proposition 4.2.1. *The braid group $\mathcal{B}_n(L_k)$ is isomorphic to a free group on*

$$1 - \frac{(n+k-2)!(2n+k-2nk-1)}{n!(k-1)!}$$

generators.
The pure braid group $\mathcal{P}_n(L_k)$ is isomorphic to a free group on

$$1 - \frac{(n+k-2)!(2n+k-2nk-1)}{(k-1)!}$$

generators.

Proof. The braid group $\mathcal{B}_n(L_k)$ is a free group since $\mathcal{Q}_n(L_k)$ is a graph. Then, in order to compute the number of generators we need to find the Euler characteristic of $\mathcal{Q}_n(L_k)$.
We follow a reasoning similar to that used in $\mathcal{Q}_n(T_k)$ in order to describe and count the 0-cells and the 1-cells of $\mathcal{Q}_n(L_k)$.
The 0-cells of $\mathcal{Q}_n(L_k)$ are the configurations satisfying one of the following characteristics:

i) one point lays on v_0 and the other $n-1$ points are distributed in the interiors of the k loops $e_1, \dots, e_k$;

ii) all the n points are distributed in the interiors of the k loops $e_1, \dots, e_k$.

Then, we count the vertices satisfying $i)$ as combinations with repetitions of $n-1$ elements and hence they are exactly $\binom{n+k-2}{n-1}$. Similarly we count the vertices satisfying $ii)$ as combinations with repetitions of n elements and hence they are exactly $\binom{n+k-1}{n}$. So, we have that the number of 0-cells of $\mathcal{Q}_n(L_k)$ is

$$\binom{n+k-2}{n-1} + \binom{n+k-1}{n}.$$

Notice that this number coincide with that of $\mathcal{Q}'_n(T_k)$.

The 1-cells of $\mathcal{Q}_n(L_k)$ are the oriented edges from a vertex of type *ii*) to a vertex of type *i*). Remind that for each 0-cell having points on a loop there are two distinct 1-cells, hence the number of 1-cells of $\mathcal{Q}_n(L_k)$ is twice the number of 1-cells of $\mathcal{Q}'_n(T_k)$.
Finally, the number of generators is equal to

$$\begin{aligned} 1-\chi(\mathcal{Q}_n(L_k)) =& 1-\left(\binom{n+k-2}{n-1}+\binom{n+k-1}{n}-2k\binom{n+k-2}{n-1}\right) \\ =& 1-\frac{(n+k-2)!(2n+k-2nk-1)}{n!(k-1)!} \end{aligned}$$

To get the number of generators for $\mathcal{P}_n(L_k)$ it is sufficient to compute $\chi(\widetilde{\mathcal{Q}}_n(L_k))$ by multiplying $\chi(\mathcal{Q}_n(L_k))$ by a factor $n!$. □

In particular, the braid group $\mathcal{B}_n(L_2)$ is isomorphic to a free product on $2n$ generators, $\mathcal{B}_n(L_3)$ is isomorphic to a free product on $n(2n+1)$ generators and $\mathcal{B}_n(L_4)$ is isomorphic to a free product on $\frac{2n^3+5n^2+n}{2}$ generators.

Let us see some examples.

The graph L_2 consists of two loops e_0 and e_1 and a single vertex v_0.
For $n=1$, we have three 0-cells and four 1-cells as in Figure 4.2.13. Then, $\pi_1(\mathcal{Q}_1(L_2)) = \langle x_1, x_2 \rangle$ and so $\mathcal{B}_1(L_2) \cong \mathbb{F}_2$.

For $n=2$, we have five 0-cells and eight 1-cells as in Figure 4.2.14. Then,

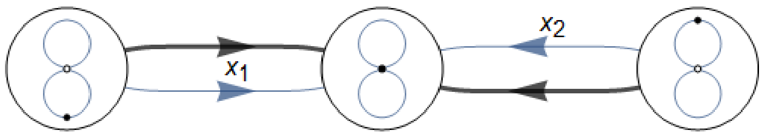

Figure 4.2.13: $\mathcal{Q}_1(L_2)$

observe that a presentation for $\pi_1(\mathcal{Q}_2(L_2))$ has four generators. Hence, $\mathcal{B}_2(L_2) \cong \mathbb{F}_4$.

The graph L_3 consists of 3 loops and a single vertex v_0.

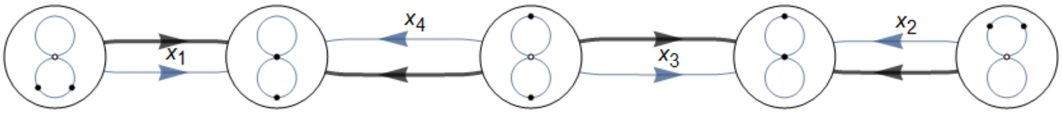

Figure 4.2.14: $\mathcal{Q}_2(L_2)$

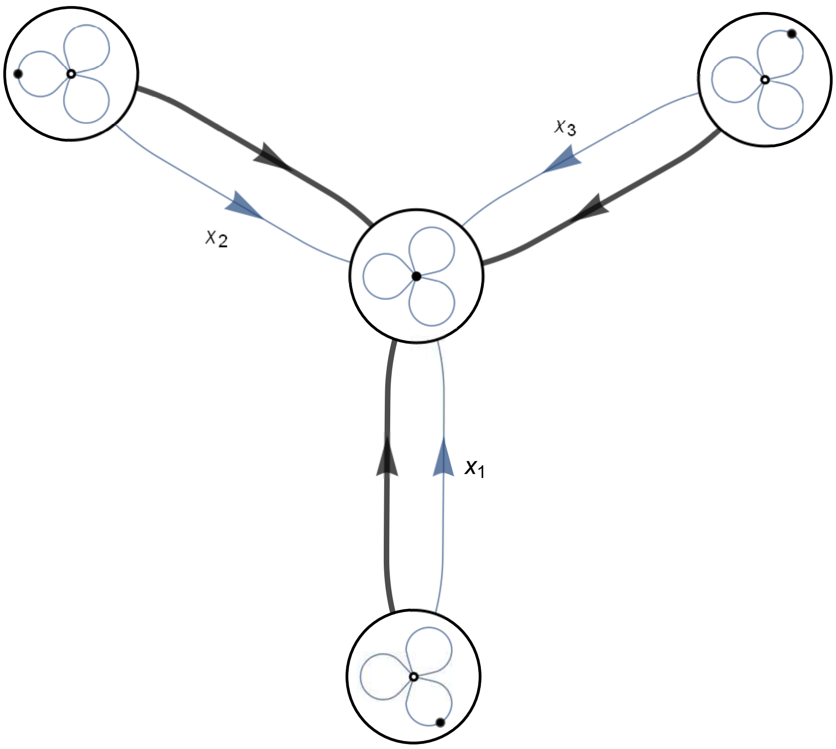

Figure 4.2.15: $\mathcal{Q}_1(L_3)$

For $n = 1$, we have four 0-cells and six 1-cells arranged as in Figure 4.2.15. Hence, a presentation for $\pi_1(\mathcal{Q}_1(L_3))$ consists of 3 generators, that is $\mathcal{B}_1(L_3) \cong \mathbb{F}_3$.

For $n = 2$, we have a presentation for $\pi_1(\mathcal{Q}_2(L_3))$ with 10 generators. Hence, $\mathcal{B}_2(L_3) \cong \mathbb{F}_{10}$.

For $n = 3$, we have a presentation for $\pi_1(\mathcal{Q}_3(L_3))$ with 21 generators. Hence, $\mathcal{B}_3(L_3) \cong \mathbb{F}_{21}$.

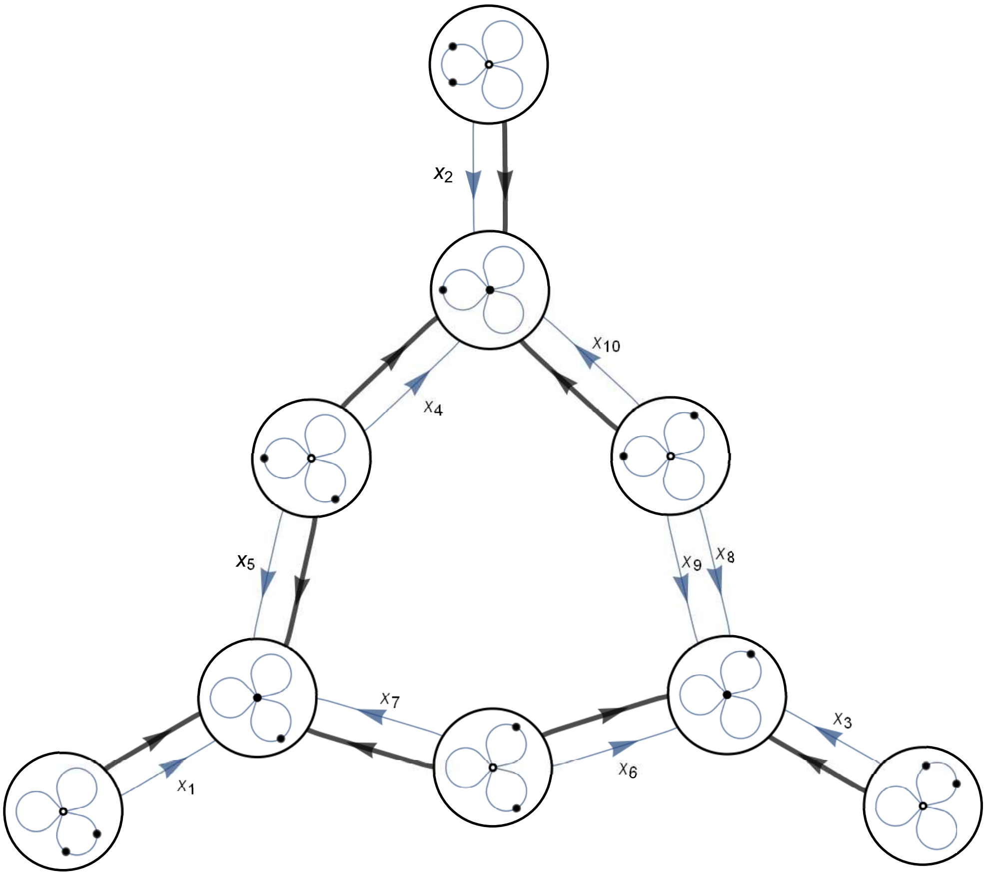

Figure 4.2.16: $\mathcal{Q}_2(L_3)$

The graph L_4 is formed by four loops and a single vertex.

For $n = 1$, we have a presentation for $\pi_1(\mathcal{Q}_1(L_4))$ with 4 generators. Hence, $\mathcal{B}_1(L_4) \cong \mathbb{F}_4$.

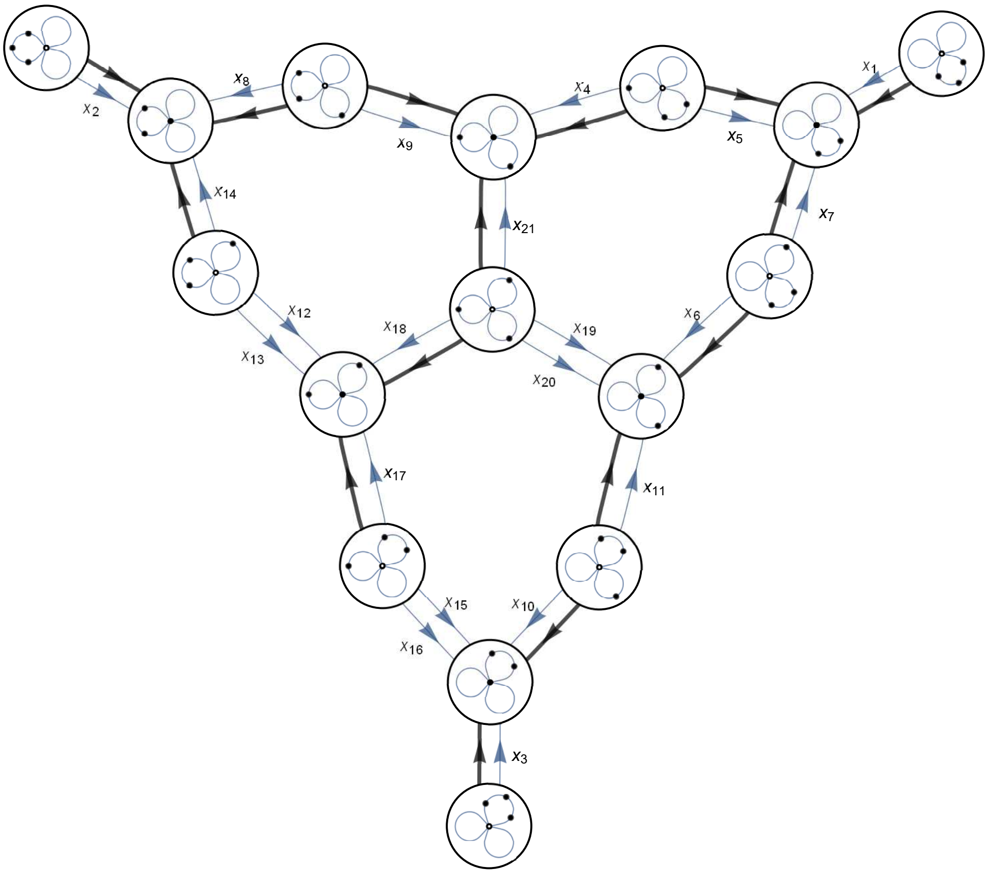

Figure 4.2.17: $\mathcal{Q}_3(L_3)$

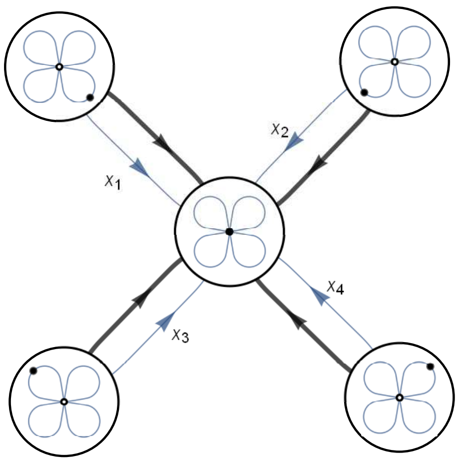

Figure 4.2.18: $\mathcal{Q}_1(L_4)$

For $n = 2$, we have a presentation for $\pi_1(\mathcal{Q}_2(L_4))$ with 19 generators. Hence, $\mathcal{B}_2(L_4) \cong \mathbb{F}_{19}$.

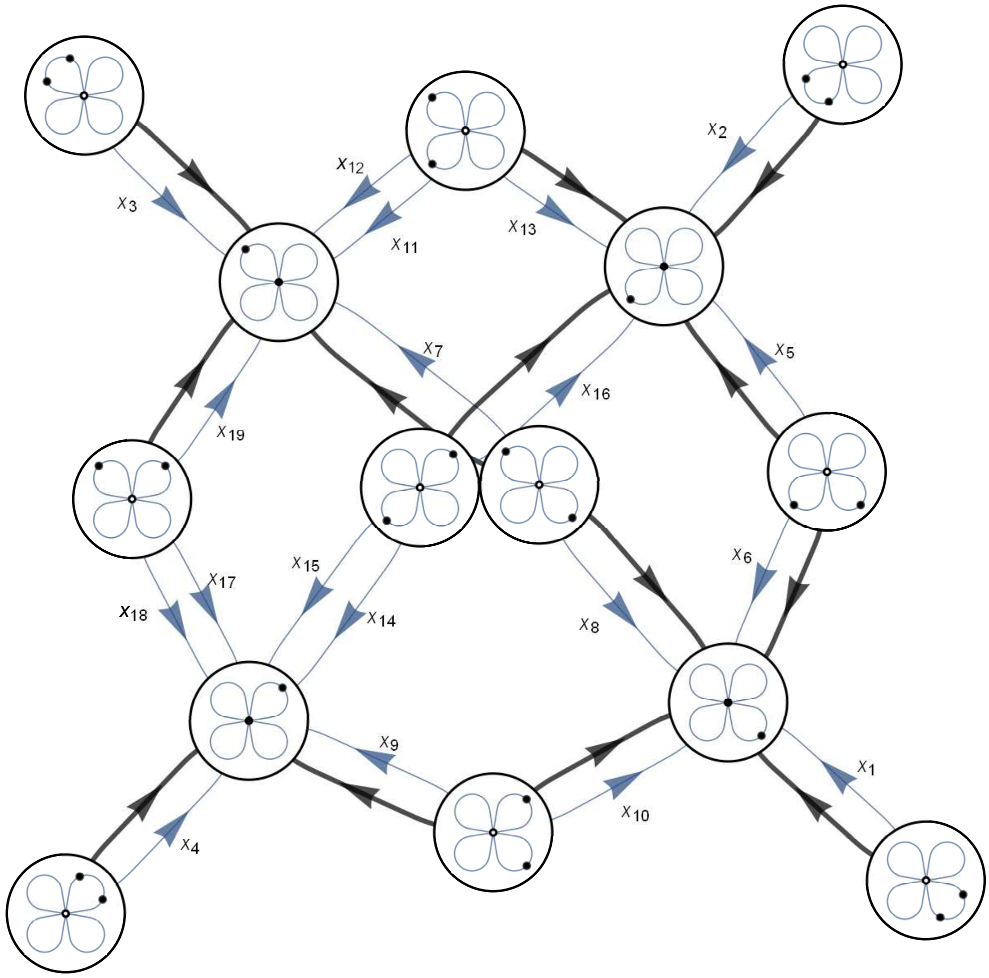

Figure 4.2.19: $\mathcal{Q}_2(L_4)$

4.3 Graphs $G_{k,h}$

Let $G_{k,h}$ with $k, h \geq 1$ be a graph consisting of k terminal vertices $v_1, \ldots, v_k$, a single vertex v_0 of degree $2h + k$, k edges such that each e_i joins v_0 and v_i and h loops attached on v_0.
By Corol. 3.3.6, the dimension of $\mathcal{Q}'_n(G_{k,h})$ is equal to 1 since there is one single vertex of degree greater than 2.

Proposition 4.3.1. *The braid group $\mathcal{B}_n(G_{k,h})$ is isomorphic to a free group on*

$$1 - \frac{(n+h+k-2)!(2n+h+k-1)}{n!(h+k-1)!} + \frac{1}{2}\Big[(2h+k)\binom{n+h+k-2}{n-1} + \sum_{k'=0}^{k}\sum_{h'=0}^{h}(2h'+k')\binom{k}{k'}\binom{h}{h'}\binom{n-1}{n-k'-h'}\Big]$$

generators.
The pure braid group $\mathcal{P}_n(G_{k,h})$ is isomorphic to a free group on

$$1 - \frac{(n+h+k-2)!(2n+h+k-1)}{(h+k-1)!} + \frac{n!}{2}\Big[(2h+k)\binom{n+h+k-2}{n-1} + \sum_{k'=0}^{k}\sum_{h'=0}^{h}(2h'+k')\binom{k}{k'}\binom{h}{h'}\binom{n-1}{n-k'-h'}\Big]$$

generators.

Proof. The braid group $\mathcal{B}_n(G_{k,h})$ is a free group since $\mathcal{Q}'_n(G_{k,h})$ is a graph. Then, in order to compute the number of generators we need to find the Euler characteristic of $\mathcal{Q}'_n(G_{k,h})$.
We follow a reasoning similar to that used in the previous two sections in order to describe and count the 0-cells and the 1-cells of $\mathcal{Q}'_n(G_{k,h})$.
The 0-cells of $\mathcal{Q}'_n(G_{k,h})$ are the configurations satisfying one of the following characteristics:

i) one point lays on v_0 and the other $n-1$ points are distributed in the interiors of the $k+h$ edges of $G_{k,h}$;

ii) all the n points are distributed in the interiors of the $k+h$ edges of $G_{k,h}$.

Then, we count the vertices satisfying *i*) as $(k+h-1)$-combinations with repetitions of $n-1$ elements and hence they are exactly $\binom{n+h+k-2}{n-1}$. Notice that these 0-cells must have degree $2h+k$. Similarly we count the vertices satisfying *ii*) as $(k+h)$-combinations with repetitions of n elements and hence they are exactly $\binom{n+h+k-1}{n}$. In this case, the degree of each 0-cell varies depending on the number of non-empty edges of $G_{k,h}$. We have that the total number of 0-cells of $\mathcal{Q}'_n(G_{k,h})$ is

$$\binom{n+h+k-2}{n-1} + \binom{n+h+k-1}{n}.$$

The 1-cells of $\mathcal{Q}'_n(G_{k,h})$ are the oriented edges from a vertex of type *ii*) to a vertex of type *i*).
The number of 1-cells of $\mathcal{Q}'_n(G_{k,h})$ is

$$\frac{1}{2}\left((2h+k)\binom{n+h+k-2}{n-1} + \sum_{k'=0}^{k}\sum_{h'=0}^{h}(2h'+k')\binom{k}{k'}\binom{h}{h'}\binom{n-1}{n-k'-h'}\right).$$

Then, the number of generators for $\mathcal{B}_n(G_{k,h})$ is equal to $1-\chi(\mathcal{Q}_n(G_{k,h}))$ and the number of generators for $\mathcal{P}_n(G_{k,h})$ is equal to $1-n!\chi(\mathcal{Q}_n(G_{k,h}))$.
□

In particular if $h=0$ or $k=0$ we obtain again the formulas already seen in the cases of T_k and L_k respectively.
If $h=k=1$, we have that $\mathcal{B}_n(G_{1,1}) \cong \mathbb{F}_n$, if $k=2$ and $h=1$ we get $\mathcal{B}_n(G_{2,1}) \cong \mathbb{F}_{n^2}$.

Now we see some examples.
The graph $G_{1,1}$ consists of two edges one of which is a loop as in Figure 4.3.20.

For $n=1$, we construct the subcomplex $\mathcal{Q}'_1(G_{1,1})$ and, as soon as we choose a maximal tree, we observe from Figure 4.3.21 that there is a single edge out of it. Hence, $\pi_1(\mathcal{Q}'_1(G_{1,1})) \cong \mathbb{F}_1$.

For $n=2$, we get two edges not contained in the maximal tree chosen, hence $\pi_1(\mathcal{Q}'_2(G_{1,1})) \cong \mathbb{F}_2$.

For $n=3$, we get three edges not contained in the maximal tree chosen and so this agrees with Prop. 4.4.1

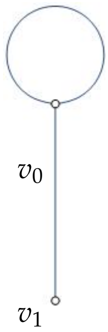

Figure 4.3.20: *Graph* $G_{1,1}$.

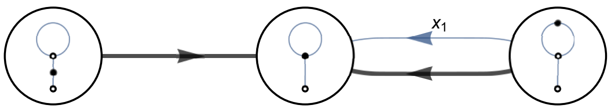

Figure 4.3.21: $\mathcal{Q}'_1(G_{1,1})$

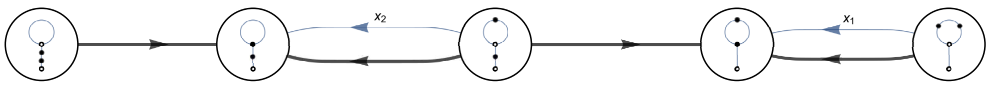

Figure 4.3.22: $\mathcal{Q}'_2(G_{1,1})$

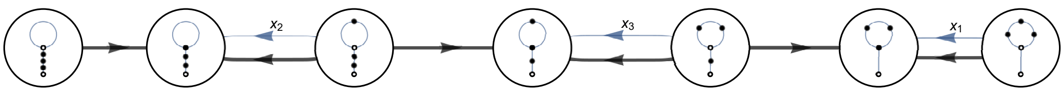

Figure 4.3.23: $\mathcal{Q}'_3(G_{1,1})$

Consider the graph $G_{2,1}$ in Figure 4.3.24. Then, $\mathcal{B}_2(G_{2,1}) \cong \mathbb{F}_4$, $\mathcal{B}_3(G_{2,1}) \cong \mathbb{F}_9$ as we see in the following figures representing the subcomplexes $\mathcal{Q}'_2(G_{2,1})$ and $\mathcal{Q}'_3(G_{2,1})$ respectively.

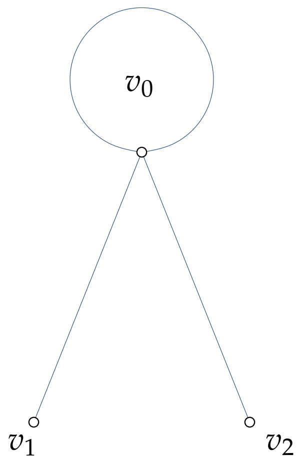

Figure 4.3.24: *Graph* $G_{2,1}$.

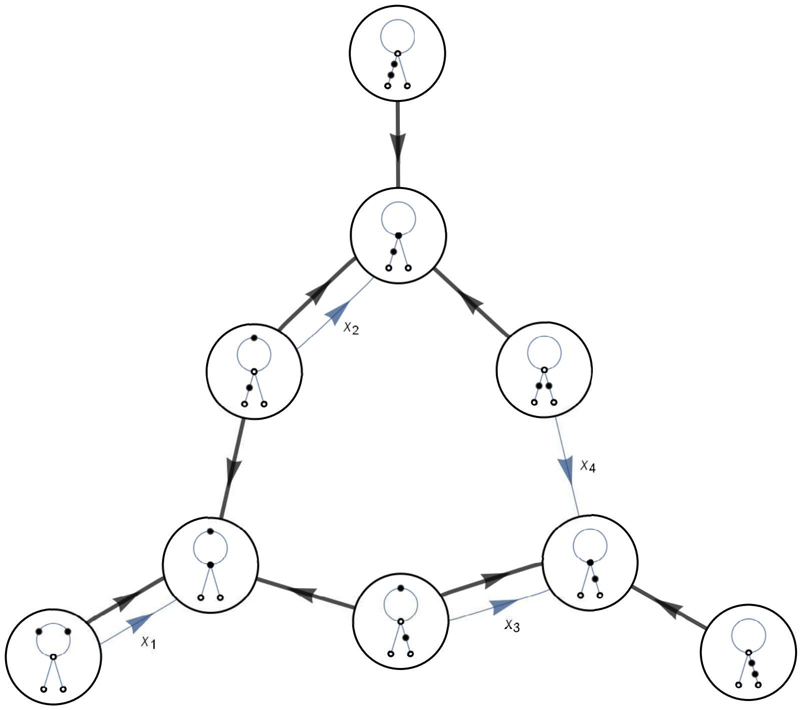

Figure 4.3.25: $\mathcal{Q}'_2(G_{2,1})$

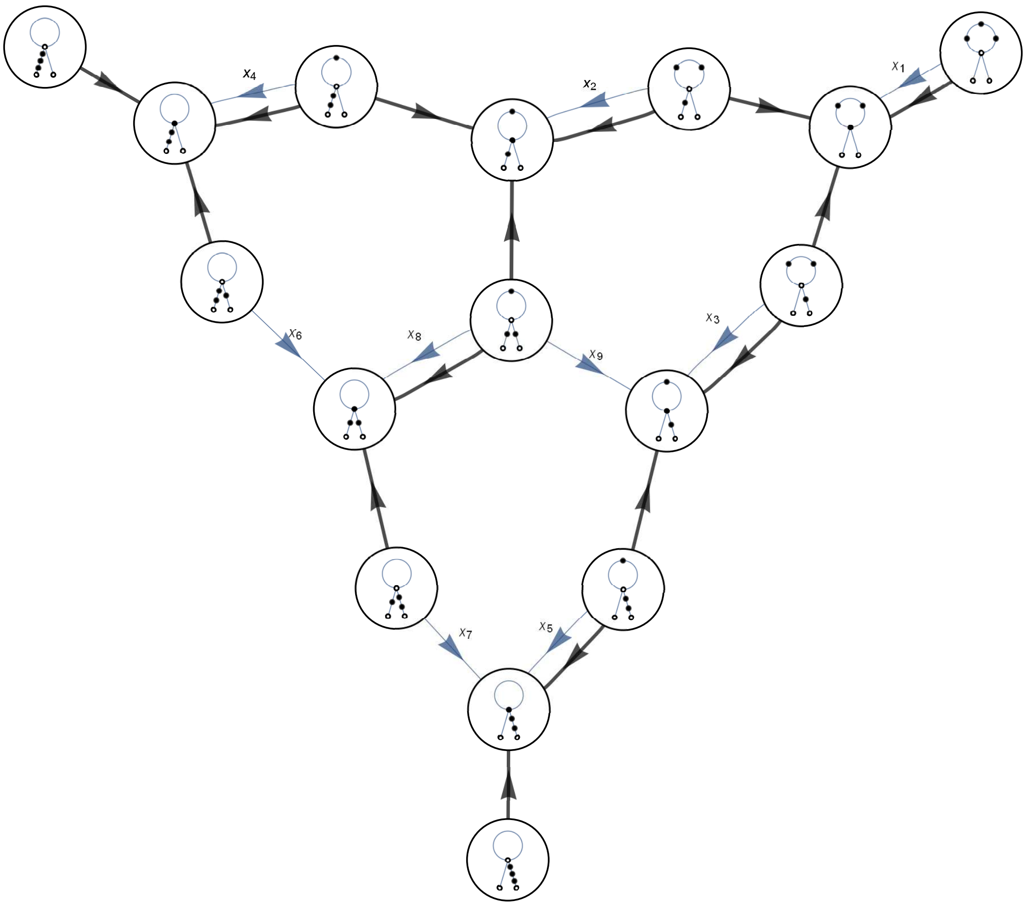

Figure 4.3.26: $\mathcal{Q}'_3(G_{2,1})$

Let us consider the graph $G_{2,2}$, then we get the following results for the braid group $\mathcal{B}_n(G_{2,2})$:

1. $\mathcal{B}_1(G_{2,2}) \cong \mathbb{F}_2$,
2. $\mathcal{B}_2(G_{2,2}) \cong \mathbb{F}_{11}$,
3. $\mathcal{B}_3(G_{2,2}) \cong \mathbb{F}_{31}$,
4. $\mathcal{B}_4(G_{2,2}) \cong \mathbb{F}_{66}$.

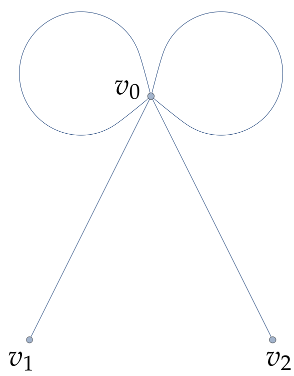

Figure 4.3.27: *Graph $G_{2,2}$.*

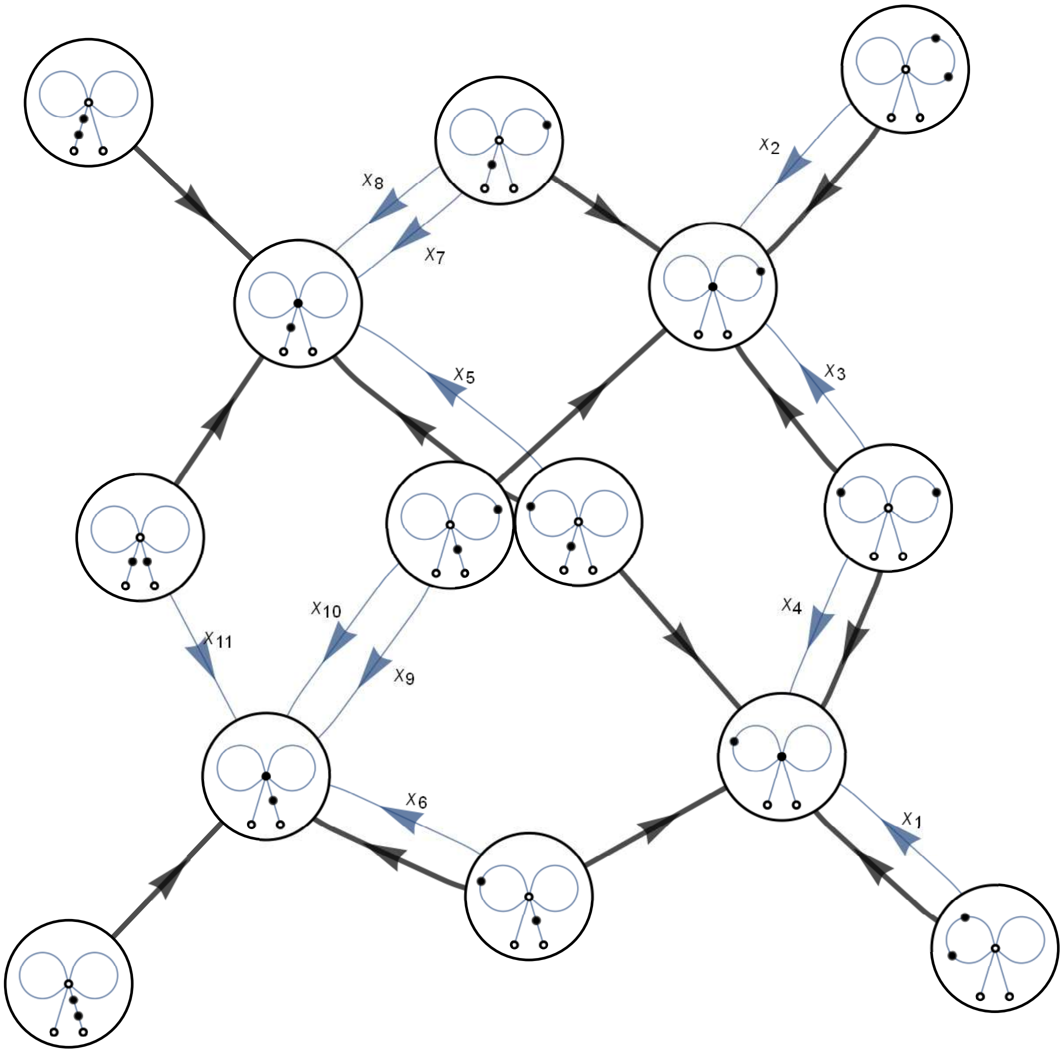

Figure 4.3.28: $\mathcal{Q}'_2(G_{2,2})$

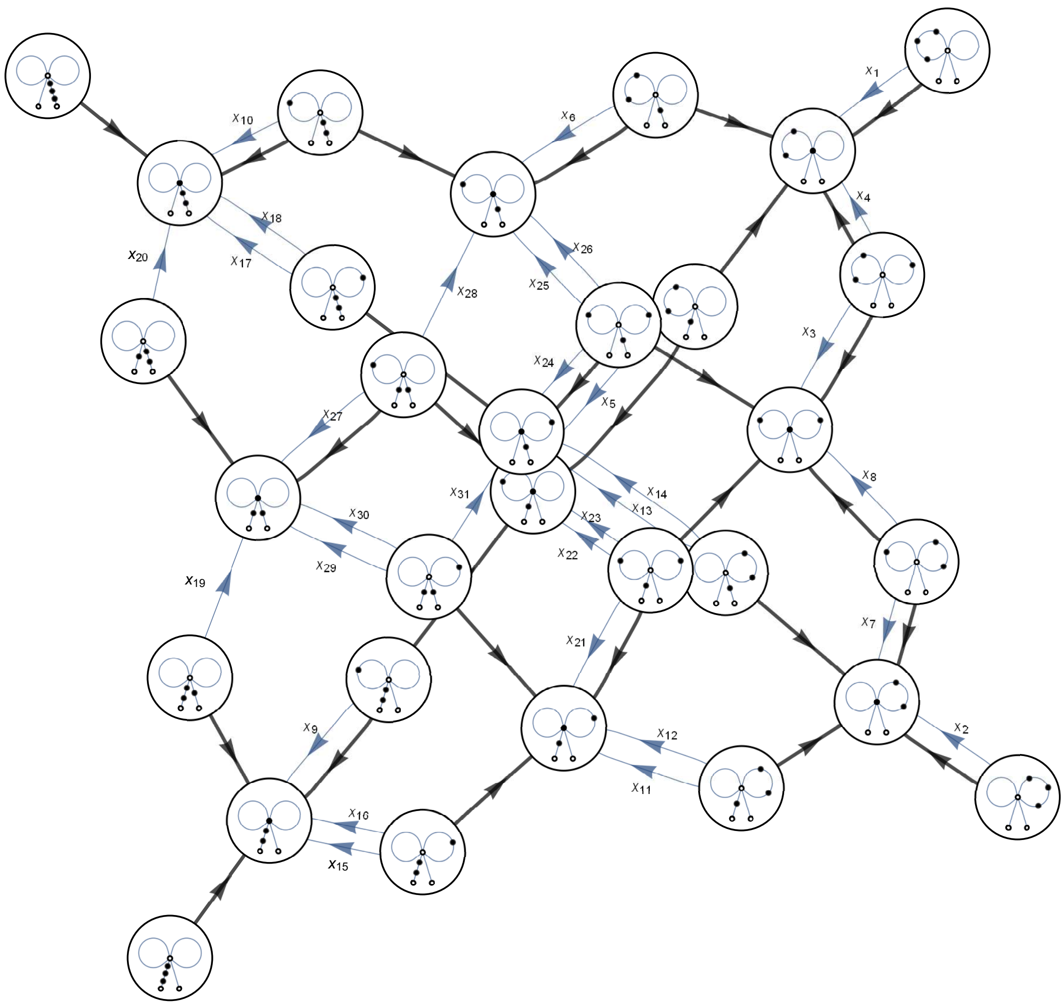

Figure 4.3.29: $\mathcal{Q}'_3(G_{2,2})$

4.4 Balloons B_k

We call *balloons* the graphs described in this section and we denote them by B_k where k is the number of loops contained in each graph. This collection of graphs is already discussed in [FS09] where it is given the following result.

Proposition 4.4.1. *[FS09] The braid group $\mathcal{B}_3(B_k)$ has $3k + 3\binom{k}{2} + 2\binom{k}{3}$ generators and $k^2 - k + (k+1)\binom{k}{2}$ relations, all of which are commutators.*

Notice that B_1 is the graph $G_{1,1}$ already seen in the previous section.
Let B_2 be the graph in Figure 4.4.30.
For $n = 2$, we get the following presentation:

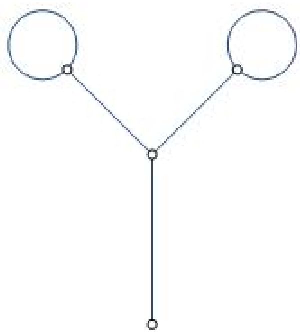

Figure 4.4.30: *Graph B_2.*

$$\pi_1(\mathcal{Q}'_1(B_2)) = \langle x_1, \ldots, x_5 | x_3^{-1}x_4^{-1}x_3x_5^{-1}x_3^{-1}x_4x_3x_5 \rangle.$$

Notice that after applying the substitution $x_4 \to x_3x_4x_3^{-1}$ the relation becomes $x_4^{-1}x_5^{-1}x_4x_5$ which is a commutator of generators x_4 and x_5. Hence, after renumbering the generators we have

$$\pi_1(\mathcal{Q}'_2(B_2)) = \langle x_1, \ldots, x_4 | [x_3, x_4] \rangle$$

which is a right-angled Artin group and in particular it is isomorphic to $\mathbb{F}_3 * \mathbb{Z}^2$. This result agrees with the corresponding one in [FS09].
For $n = 3$, we get a presentation for $\mathcal{B}_3(B_2)$ with 9 generators and 6

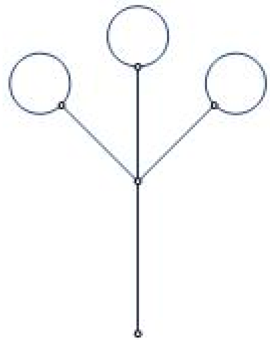

Figure 4.4.31: *Graph B_3.*

relations. By Prop. 4.4.1, we should have 9 generators and 5 relations and indeed after a substitution we observe that two of the found relations are equivalent.

Let us consider the graph B_3 in Figure 4.4.31.
For $n = 2$, we get the following presentation:

$$\pi_1(\mathcal{Q}'_2(B_3)) = \langle x_1, \ldots, x_9 \mid x_4^{-1}x_7^{-1}x_4x_8^{-1}x_4^{-1}x_7x_4x_8,\\ x_5^{-1}x_7^{-1}x_5x_9^{-1}x_5^{-1}x_7x_5x_9,\ x_6^{-1}x_8^{-1}x_6x_9^{-1}x_6^{-1}x_8x_6x_9\rangle.$$

By applying the substitutions $x_7 \to x_4x_7x_4^{-1}$, $x_9 \to x_5^{-1}x_9x_5$ and $x_8 \to x_6x_8x_6^{-1}$ then the relations become commutators of generators: $x_7x_8x_7^{-1}x_8^{-1}$, $x_7x_9x_7^{-1}x_9^{-1}$ and $x_8x_9x_8^{-1}x_9^{-1}$. Then, after renumbering the generators we get

$$\mathcal{B}_2(B_3) = \langle x_1, \ldots, x_6 | [x_4, x_5], [x_5, x_6], [x_4, x_6]\rangle.$$

Hence, $\mathcal{B}_2(B_3)$ is a right-angled Artin group and it is isomorphic to $\mathbb{F}_6 * \mathbb{Z}^3$.

4.5 Graphs Θ_k

Let Θ_k be the graph consisting of two vertices and k edges such that each edge joins the two vertices.
For $k = 1$, Θ_1 consists of a single edge and its endpoints, hence $\mathcal{Q}_n(\Theta_1)$ is contractible since dim $\mathcal{Q}'_n(\Theta_1) = 0$.
For $k = 2$, the graph Θ_2 is the graph in Figure 4.5.32 and we observe that the results agree with those already seen for L_1.

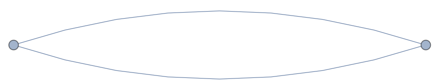

Figure 4.5.32: *Graph Θ_2.*

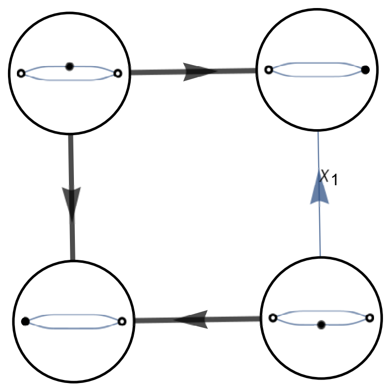

Figure 4.5.33: *The subcomplex $\mathcal{Q}'_1(\Theta_2)$.*

For $k = 3$, we have that Θ_3 is homeomorphic to the capital greek letter Θ. Then, we get $\mathcal{B}_2(\Theta_3) \cong \mathcal{B}_3(\Theta_3) \cong \mathbb{F}_3$.

Let us consider the graph Θ_4.
We get $\mathcal{B}_1(\Theta_4) \cong \mathbb{F}_3$ and $\mathcal{B}_2(\Theta_4) \cong \mathbb{F}_6$.

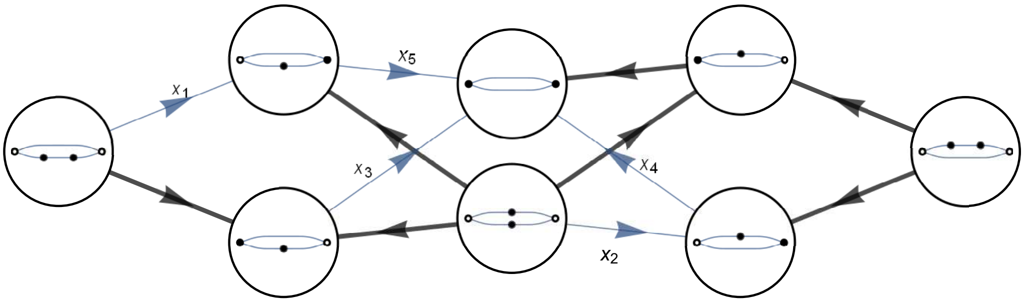

Figure 4.5.34: *The subcomplex* $\mathcal{Q}_2'(\Theta_2)$.

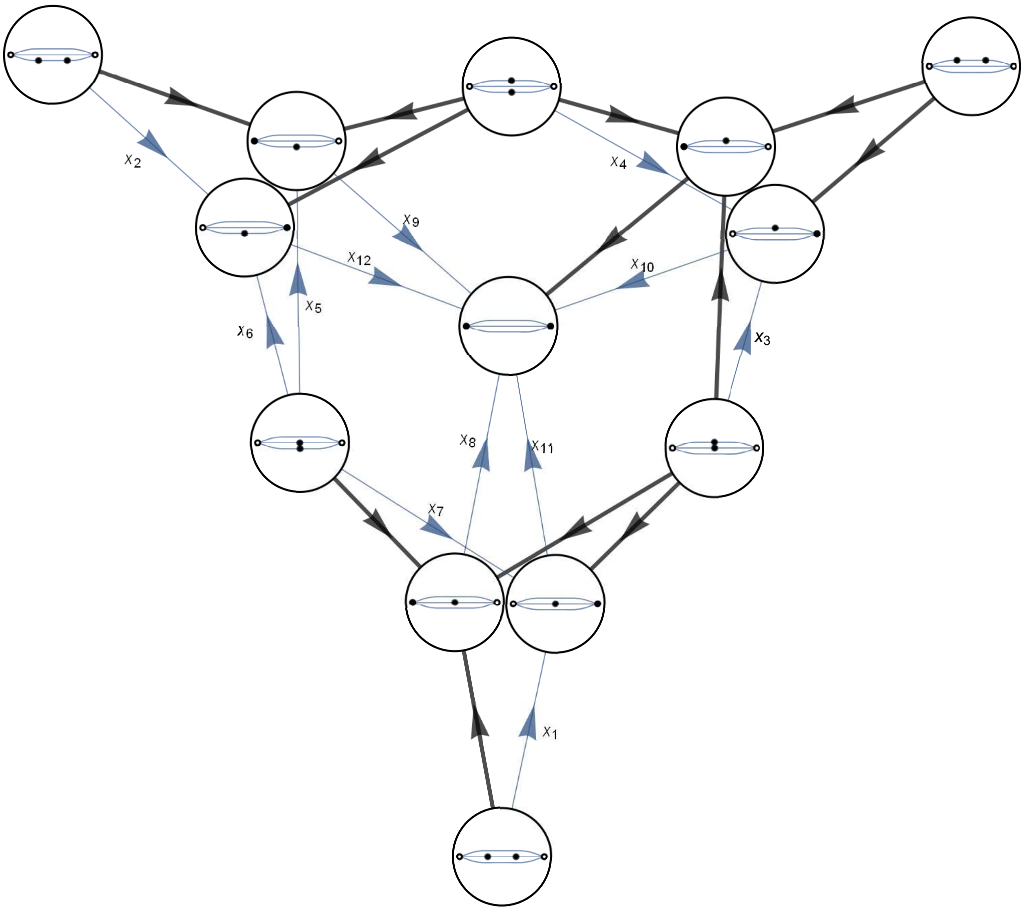

Figure 4.5.35: $\mathcal{Q}_2'(\Theta_3)$.

4.6 Chains C_k

Let C_k with $k \geq 3$ indicate the graph consisting of k vertices and $2(k-1)$ edges such that each pair of vertices is joined by two edges as a chain.
Let C_3 be the graph in Figure 4.6.36.
Notice that the results for C_3 coincide to those for L_2. Let C_4 be the graph

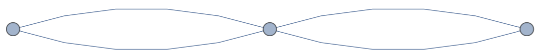

Figure 4.6.36: *Graph* C_3.

in Figure 4.6.37.
For $n = 1$, we can see from Figure 4.6.38 that there are three generators for the braid group $\mathcal{B}_1(C_4)$.
For $n = 2$, we get the following presentation

$$\mathcal{B}_2(C_4) = \langle x_1, \ldots, x_7 | x_1^{-1} x_6 x_7^{-1} x_6^{-1} x_1 x_6 x_7 x_6^{-1} \rangle.$$

Then, applying the substitution $x_7 \to x_6^{-1} x_7 x_6$ and renumbering the generators we have:

$$\mathcal{B}_2(C_4) = \langle x_1, \ldots, x_6 | [x_1, x_6] \rangle$$

which is a right-angled Artin group and hence, $\mathcal{B}_2(C_4) \cong \mathbb{F}_4 * \mathbb{Z}^2$.

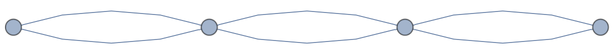

Figure 4.6.37: *Graph* C_4.

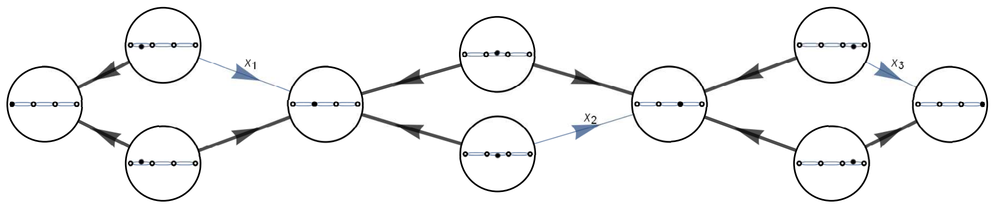

Figure 4.6.38: *The subcomplex $\mathcal{Q}'_1(C_4)$.*

4.7 The smallest non linear tree T

Let T be the smallest non linear tree as in Figure 4.7.39.
For $n = 1$, the subcomplex $\mathcal{Q}'_1(T)$ is a tree and hence the braid group

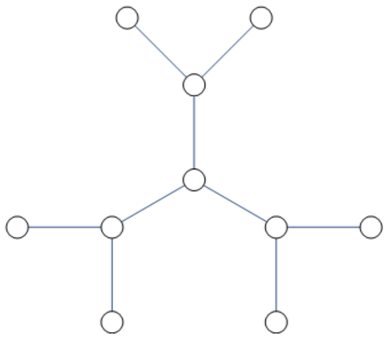

Figure 4.7.39: *Graph T.*

$\mathcal{B}_1(T)$ is trivial.
For $n = 2$, the presentation can be reduced to the form:

$$\pi_1(\mathcal{Q}'_2(T)) = \langle x_1, x_2, x_3, x_4 \rangle$$

and so $\mathcal{B}_2(T)$ is isomorphic to $\mathbb{F}_4$.

For $n = 3$, we get $\mathcal{B}_3(T) \cong \mathbb{F}_{12}$. These results agree with 2.3.11, indeed the braid group is a right-angled Artin group if $n < 4$. Now we are going to verify that for $n = 4$ we cannot get a right-angled Artin group.

For $n = 4$, we get a presentation consisting of 24 generators, 4 commutators of generators and two other relations. We observe that there are 16 generators not involved in the relations, hence the braid group $\mathcal{B}_4(T)$ is isomorphic to $\mathbb{F}_{16} * A$ where A is given by the relations. After having eliminated the free generators, we get the following reduced presentation:

$$\langle x_1, \ldots, x_8 \mid [x_2, x_6], [x_4, x_7], [x_6, x_7], [x_6, x_8], x_1^{-1}x_3x_8^{-1}x_3^{-1}x_1x_8, \\ x_1^{-1}x_5^{-1}x_7^{-1}x_5x_1x_8^{-1}x_1^{-1}x_5^{-1}x_7x_5x_1x_8\rangle.$$

After applying the substitutions $x_3 \to x_1x_3$ and $x_5 \to x_5x_1^{-1}$ we obtain:

$$\langle x_1, \ldots, x_8 \mid [x_2, x_6], [x_4, x_7], [x_6, x_7], [x_6, x_8], [x_3, x_8], x_5^{-1}x_7^{-1}x_5x_8^{-1}x_5^{-1}x_7x_5x_8\rangle$$

but it is not possible to write all the relations as commutators of generators. Hence, $\mathcal{B}_4(T)$ is not a right-angled Artin group.

4.8 Other examples with non-trivial relations

We call H the graph homeomorphic to the capital letter H.
For $n = 1$, notice that $\mathcal{Q}'_1(H)$ is a tree isomorphic to H and hence

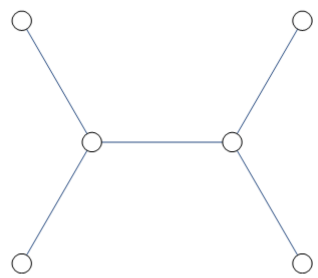

Figure 4.8.40: *Graph H.*

$\pi_1(\mathcal{Q}'_1(H))$ is trivial. For $n = 2$, we can observe in Figure 4.8.43 that

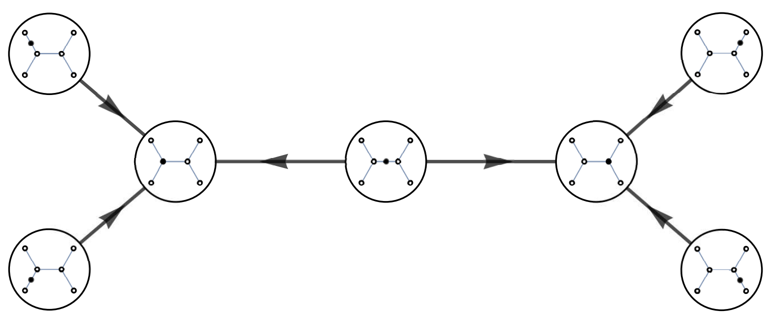

Figure 4.8.41: *The cubical subcomplex* $\mathcal{Q}'_1(H)$.

there are eleven edges out of the chosen tree and so we get $\pi_1(\mathcal{Q}'_2(H)) = \langle x_1, x_2 \rangle$. Hence, $\mathcal{B}_2(H) \cong \mathbb{F}_2$.
For $n = 3$, we get $\mathcal{B}_3(H) \cong \mathbb{F}_6$.
For $n = 4$, we have:

$$\pi_1(\mathcal{Q}'_4(H)) = \langle x_1, \ldots, x_{12} | x_6 x_{11} x_{12}^{-1} x_6^{-1} x_{12} x_{11}^{-1} \rangle.$$

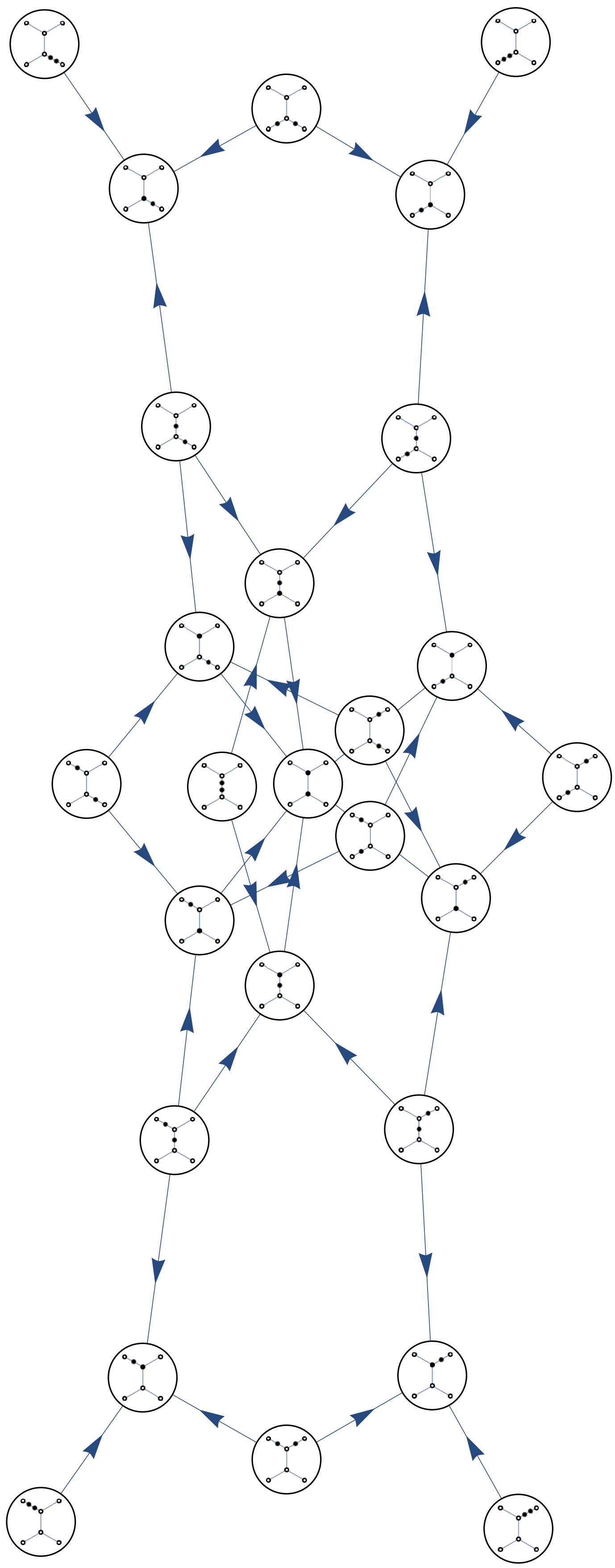

Figure 4.8.42: *The subcomplex* $\mathcal{Q}_2'(H)$.

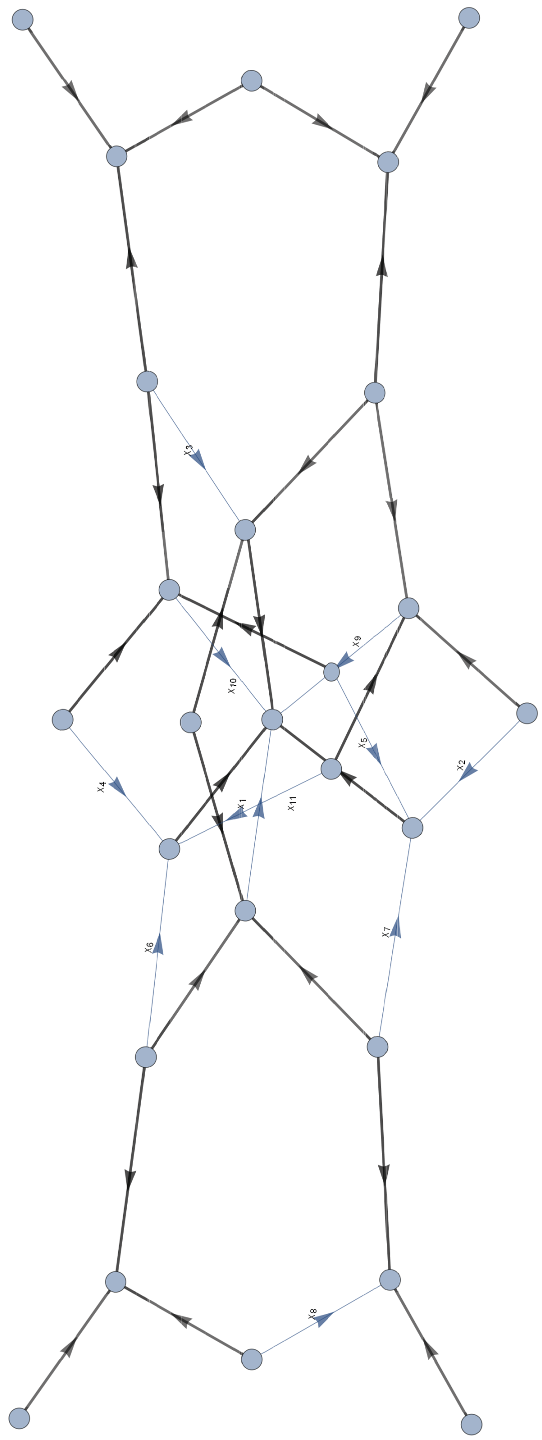

Figure 4.8.43: *The maximal tree chosen on the subcomplex* $\mathcal{Q}_2'(H)$.

If we apply the substitution $x_{11} \to x_{11}x_{12}$ the relation becomes a commutator of generators x_6 and x_{11} and then $\mathcal{B}_4(H) \cong \mathbb{F}_{10} * \mathbb{Z}^2$.
For $n = 5$, after similar substitutions and renumberings, we find that also $\mathcal{B}_5(H)$ is a right-angled Artin group consisting of 20 generators and 5 commutators.
For $n = 6$, $\mathcal{B}_6(H) \cong \mathbb{F}_{19} * (\mathbb{F}_1 \times \mathbb{F}_4) * (\mathbb{F}_1 \times \mathbb{F}_5)$.

Let us call Γ the graph consisting of two vertices of degree 3 and three edges arranged as in Figure 4.8.44. Notice that there are no terminal vertices in Γ, hence $\mathcal{Q}_n(\Gamma) = \mathcal{Q}'_n(G)$.
For $n = 1$, we can construct the cubical subcomplex $\mathcal{Q}_1(\Gamma)$ as in Fig-

Figure 4.8.44: *Graph* Γ.

ure 4.8.45. Let us observe that when we choose a maximal spanning tree, then there are 2 edges out of it. Hence, $\pi_1(\mathcal{Q}_1(\Gamma)$ is isomorphic to $\mathbb{F}_2$.
For $n = 2$, a presentation for $\pi_1(\mathcal{Q}_2(\Gamma))$ is

$$\pi_1(\mathcal{Q}_2(\Gamma) = \langle x_1, \ldots, x_4 | [x_3, x_4] \rangle.$$

Hence, $\mathcal{B}_2(\Gamma) \cong \mathbb{F}_2 * \mathbb{Z}^2$.
For $n = 3$, a presentation for $\pi_1(\mathcal{Q}_3(\Gamma))$ is

$$\mathcal{B}_3(\Gamma) = \langle x_1, \ldots, x_6 | [x_3, x_6], [x_4, x_6], [x_4, x_5] \rangle$$

and hence $\mathcal{B}_3(\Gamma) \cong \mathbb{F}_2 * \mathbb{Z}^4$.
For $n = 4$, the braid group $\mathcal{B}_4(\Gamma)$ is a right-angled Artin group indeed a

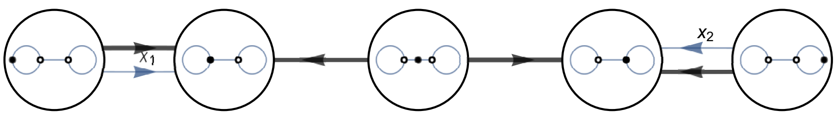

Figure 4.8.45: *The cubical complex* $\mathcal{Q}_1(\Gamma)$.

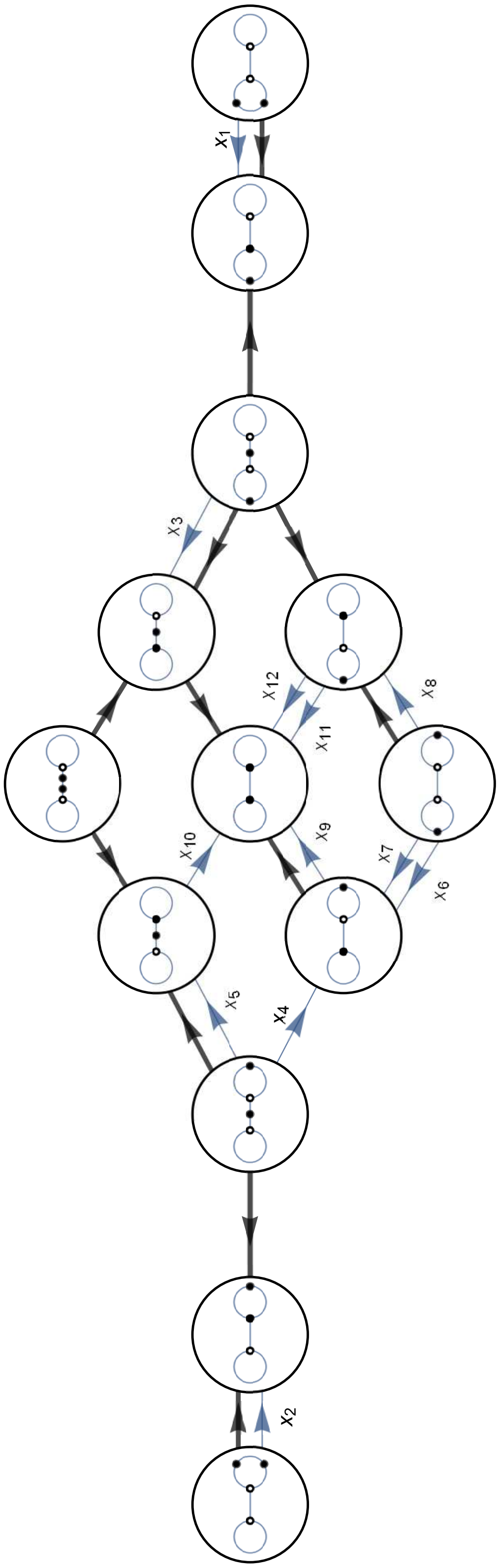

Figure 4.8.46: *The subcomplex* $\mathcal{Q}_2(\Gamma)$.

presentation is the following

$$\mathcal{B}_4(\Gamma) = \langle x_1, \ldots, x_8 | [x_3, x_8], [x_4, x_7], [x_4, x_8], [x_5, x_6], [x_5, x_7], [x_5, x_8] \rangle.$$

Appendix A

Mathematica code

We see in detail the code used to create figures and to compute presentations for braid groups as seen in chapter 4.

Given a graph G we want to construct the corresponding cubical complex $\mathcal{Q}_n(G)$ and the subcomplex $\mathcal{Q}'_n(G)$.
The function *ConfigVertices* takes in input the graph G, the number of points n and the type of the cubical complex chosen: nothing if we want to consider $\mathcal{Q}_n(G)$, "Reduced" if we want $\mathcal{Q}'_n(G)$. The outputs are all the 0-cells of $\mathcal{Q}_n(G)$ or $\mathcal{Q}'_n(G)$ respectively.
ConfigEdges takes the graph G and the list of 0-cells found and give us the 1-cells as edges between the 0-cells and then *ConfigGraph* combine these two classes of cells into a unique structure.

```
BWDisk[c_] :=
 Graphics[{Opacity[1], EdgeForm[{AbsoluteThickness[1], Black}],
   If[c === 0, White, Black], Disk[]}]

ConfigVertices[g_Graph, n_Integer, type : (Reduced | Null) : Null] :=
 Module[{tg = EdgeTaggedGraph[g], nv, ne, vd, cv},
  nv = VertexCount[tg]; ne = EdgeCount[tg];
  vd = VertexDegree[tg];
  cv = Flatten[Table[Outer[List,
       Flatten[Map[Permutations, IntegerPartitions[m, {nv}, Range[0, m]]], 1],
       Flatten[Map[Permutations, IntegerPartitions[n - m, {ne},
          Range[0, n - m]]], 1], 1], {m, 0, n}], 2];
  Switch[type,
   Null, Cases[cv, {c_, _} /; Max[c] ≤ 1],
   Reduced, Cases[cv, {c_, _} /; And[Max[c] ≤ 1, c.(1 - Sign[vd - 1]) == 0]],
   Extended, Cases[cv, {c_, _} /; Max[c - vd] ≤ 0]]]
```

```
ConfigEdges[g_Graph, cv_List] :=
 Module[{tg = EdgeTaggedGraph[g], lv, le, nv, ne, ncv, l},
  lv = VertexList[tg]; le = EdgeList[tg];
  nv = VertexCount[tg]; ne = EdgeCount[tg]; ncv = Length[cv];
  l = Flatten[Table[If[cv[[i, 2, j]] > 0,
      {{i, j, Position[lv, le[[j, 1]], 1][[1, 1]]},
       {i, j, Position[lv, le[[j, 2]], 1][[1, 1]]}},
      Nothing], {i, 1, ncv}, {j, 1, ne}], 2];
  l = Replace[l, {i_, j_, k_} :→ {i, cv[[i]] + {UnitVector[nv, k], -UnitVector[ne, j]}},
    {1}];
  Cases[l, {i_, j_} /; MemberQ[cv, j] :→ DirectedEdge[i, Position[cv, j, 1][[1, 1]]]]]

ConfigEdges[g_Graph, n_Integer, type : (Extended | Reduced | Null) : Null] :=
 ConfigEdges[g, ConfigVertices[g, n, type]]

ConfigGraph[g_Graph, cv_List, ce_List] :=
 Graph[Range[Length[cv]], ce, VertexWeight → Apply[v, cv, {1}]]
ConfigGraph[g_Graph, cv_List] := ConfigGraph[g, cv, ConfigEdges[g, cv]]
ConfigGraph[g_Graph, n_Integer, type : (Extended | Reduced | Null) : Null] :=
 ConfigGraph[g, ConfigVertices[g, n, type]]
ConfigGraphPlot[cg_Graph, g_Graph, n_Integer, type : (Reduced | Null) : Null,
  opts___Rule] :=
 ConfigGraphPlot[cg, g, ConfigVertices[g, n, type], opts]
ConfigGraphPlot[g_Graph, n_Integer, type : (Reduced | Null) : Null,
  opts___Rule] :=
 ConfigGraphPlot[ConfigGraph[g, n, type], g, n, type, opts]

ConfigGraphPlot[cg_Graph, g_Graph, cv_List, opts___Rule] :=
 Module[{cw = Replace[ConfigView, Flatten[{opts, ConfigView → True}]],
         cs = Replace[ConfigSize, Flatten[{opts, ConfigSize → 50}]],
         cz = Replace[ConfigZoom, Flatten[{opts, ConfigZoom → 1}]],
         ps = Replace[PointSize, Flatten[{opts, PointSize → 0.025}]],
         ap = Replace[ArrowPlacement, Flatten[{opts, ArrowPlacement → 0.6}]],
         as = Replace[ArrowSize, Flatten[{opts, ArrowSize → 0.025}]],
         ls = Replace[LoopSize, Flatten[{opts, LoopSize → 1}]],
```

```
  r = Max[1, Map[Norm, Replace[VertexCoordinates, AbsoluteOptions[g]]]]},
 GraphPlot[cg,
  VertexLabels → If[Not[cw], All,
    Thread[Rule[VertexList[cg],
      Map[
       Placed[Show[Graphics[{EdgeForm[{AbsoluteThickness[1], Black}],
            White, Disk[{0, 0}, 1.25 r / cz]}],
          ConfigPlot[g, #, SelfLoopStyle → ls], ImageSize → cs,
          PointSize → ps], Center] &, cv]]]],
  EdgeShapeFunction → Function[{pts, edge},
    {If[Max[cv[[{edge[[1]], edge[[2]]}, 1]]] > 1, Red],
                                          Opacity[1],
     Arrowheads[{{as, ap}}], Arrow[pts]}],
  Apply[Sequence, FilterRules[Flatten[{opts}], Options[GraphPlot]]]]]
```

The function *AnnotatedConfigGraph* chooses a spanning tree on the graph resulted by *ConfigGraph* and associates a numbering to the edges which are left out of the spanning tree.

```
AnnotatedConfigGraph[cg_Graph, t_Graph] :=
 Module[{tcg = EdgeTaggedGraph[cg], tt = EdgeTaggedGraph[t], e, ne, te, k = 1},
  e = EdgeList[tcg]; ne = Length[e];
  te = ReplaceAll[EdgeList[tt], UndirectedEdge → DirectedEdge];
  Graph[VertexList[tcg], EdgeList[tcg],
   EdgeShapeFunction → Table[If[MemberQ[te, e[[i]]],
      e[[i]] → ({Black, Thick, Arrowheads[{{0.025, 0.6}}], Arrow[BSplineCurve[#1]]} &),
      e[[i]] → ({Arrowheads[{{0.025, 0.6}}], Arrow[BSplineCurve[#1]]} &)], {i, 1, ne}],
   EdgeLabels → Table[If[MemberQ[te, e[[i]]],
      e[[i]] → "", e[[i]] → Placed[Subscript[x, k++], {0.5, {0, 0}}]], {i, 1, ne}],
   VertexWeight → AnnotationValue[cg, VertexWeight]]]

AnnotatedConfigGraph[g_Graph, n_Integer, red : (Reduced | Null) : Null, opts___Rule] :=
 AnnotatedConfigGraph[ConfigGraph[g, n, red], opts]
AnnotatedConfigGraph[cg_Graph, opts___Rule] :=
 AnnotatedConfigGraph[cg, FindSpanningTree[UndirectedGraph[cg], opts]]
```

Then, to compute a presentation for the braid group $\mathcal{B}_n(G)$ we define the function *BraidGroupPresentation* which takes as input the graph obtained by using *AnnotatedConfigGraph*. The generators of the presentation

```
AnnotatedConfigGraphPlot[acg_Graph?EdgeTaggedGraphQ, g_Graph, cv_List, opts___Rule] :=
 Module[{cw = Replace[ConfigView, Flatten[{opts, ConfigView → True}]],
        cs = Replace[ConfigSize, Flatten[{opts, ConfigSize → 50}]],
        cz = Replace[ConfigZoom, Flatten[{opts, ConfigZoom → 1}]],
        ps = Replace[PointSize, Flatten[{opts, PointSize → 0.025}]],
        ap = Replace[ArrowPlacement, Flatten[{opts, ArrowPlacement → 0.6}]],
        as = Replace[ArrowSize, Flatten[{opts, ArrowSize → 0.01}]],
        ls = Replace[LoopSize, Flatten[{opts, LoopSize → 1}]],
        r = Max[1, Map[Norm, Replace[VertexCoordinates, AbsoluteOptions[g]]]],
        esf = Replace[EdgeShapeFunction, AbsoluteOptions[acg, EdgeShapeFunction]]},
  GraphPlot[acg,
   VertexLabels → If[Not[cw], All, Thread[Rule[VertexList[acg],
       Map[Placed[Show[Graphics[{EdgeForm[{AbsoluteThickness[1], Black}], White, Disk[{0, 0}, 1.25 r / cz]}],
           ConfigPlot[g, #, SelfLoopStyle → ls], ImageSize → cs, PointSize → ps], Center] &, cv]]]],
   EdgeShapeFunction → ReplaceAll[esf, Arrowheads[_] → Arrowheads[{{as, ap}}]],
   Apply[Sequence, FilterRules[Flatten[{opts}], Options[GraphPlot]]]]]

AnnotatedConfigGraphPlot[cg_Graph, t_Graph, g_Graph, cv_List, opts___Rule] :=
 AnnotatedConfigGraphPlot[AnnotatedConfigGraph[cg, t], g, cv, opts]
AnnotatedConfigGraphPlot[g_Graph, n_Integer, red : (Reduced | Null) : Null, opts___Rule] :=
 AnnotatedConfigGraphPlot[ConfigGraph[g, n, red], g, n, red, opts]
AnnotatedConfigGraphPlot[cg_Graph, g_Graph, n_Integer, red : (Reduced | Null) : Null, opts___Rule] :=
 AnnotatedConfigGraphPlot[cg, g, ConfigVertices[g, n, red], opts]
AnnotatedConfigGraphPlot[cg_Graph, g_Graph, cv_List, opts___Rule] :=
 AnnotatedConfigGraphPlot[cg, FindSpanningTree[UndirectedGraph[cg],
   Apply[Sequence, FilterRules[Flatten[{opts}], Options[FindSpanningTree]]]], g, cv, opts]
```

obtained are the edges out of the spanning tree chosen by *AnnotatedConfigGraph* and the relations are taken considering the edges which constitute the boundaries of the 2-cells of the cubical complex $\mathcal{Q}_n(G)$ or $\mathcal{Q}'_n(G)$.

Finally, we need to simplify the presentation obtained by *BraidGroupPresentation* and hence we define the function *ReducePresentation*, which applying the Tietze transformations whenever possible manages to output a reduced presentation for $\mathcal{B}_n(G)$.

```
BraidGroupPresentation[acg_Graph ? EdgeTaggedGraphQ] :=
 Module[{v = VertexList[acg], e = EdgeList[acg],
        wl = AnnotationValue[acg, VertexWeight], l, s, gen, rel},
  l = AnnotationValue[acg, EdgeLabels] /. {Placed[x_, ___] :> x, "" → 0};
  gen = Sort[DeleteCases[Map[Last, l], 0]];
  l = Thread[{e, e /. l}] /. DirectedEdge → List;
  l = Flatten[Table[Outer[List, Cases[l, {{_, i, _}, _}], Cases[l, {{i, _, _}, _}], 1], {i, v}], 2];
  l = Replace[l, {{{i_, j_, p_}, x_}, {{j_, k_, q_}, y_}} :> {i, j, k, p, q, {x, y}}, {1}];
  l = GatherBy[l, #[[{1, 3}]] &];
  rel = Apply[Join, Table[ReplaceList[s,
        {___, {i_, j1_, k_, p_, q_, w_}, ___, {i_, j2_, k_, q_, p_, v_}, ___} /;
         And[j1 =!= j2, If[wl[[j1, 2]] === wl[[j2, 2]],
                           Max[wl[[i, 2]] - wl[[k, 2]]] == 2, wl[[j1, 1]] =!= wl[[j2, 1]]]]
         :> Join[w, -Reverse[v]]], {s, l}]];
  {gen, DeleteCases[DeleteCases[rel, 0, Infinity], {}]}]

BraidGroupPresentation[cg_Graph, t_Graph] :=
 BraidGroupPresentation[AnnotatedConfigGraph[cg, t]]
BraidGroupPresentation[g_Graph, n_Integer, red : (Reduced | Null) : Null, opts___Rule] :=
 BraidGroupPresentation[ConfigGraph[g, n, red], opts]
BraidGroupPresentation[cg_Graph, opts___Rule] :=
 BraidGroupPresentation[cg, FindSpanningTree[UndirectedGraph[cg], opts]]
```

```
ReduceWord[w_List] :=
  ReplaceRepeated[DeleteCases[w, 0],
   {{a___, x_, y_, b___} /; x + y === 0 :→ {a, b},
    {x_, c___, y_} /; x + y === 0 :→ {c}}];

NormWord[w_List] :=
 Sort[Flatten[Table[{RotateLeft[w, i], RotateLeft[-Reverse[w], i]}, {i, 1, Length[w]}], 1]][[1]]

ReducePresentation[{gen_, rel_}] :=
 Module[{gm = gen, rm = Map[ReduceWord, rel]},
  Monitor[
   ReplaceRepeated[{gm, Sort[DeleteCases[Map[ReduceWord, rm], {}]]},
    {{g_, {r___, {-x_ | x_}, s___}} :→
      {gm = DeleteCases[g, x], rm = DeleteCases[Map[ReduceWord, {r, s} /. x :→ 0], {}]},
     {g_, {r___, {w___, -x_, v___}, s___}} /; FreeQ[{w, v}, x | -x] :→
      {gm = DeleteCases[g, x], rm = Sort[DeleteCases[Map[ReduceWord, {r, s} /.
            {-x :→ Apply[Sequence, -Reverse[{v, w}]], x :→ Apply[Sequence, {v, w}]}], {}]]},
     {g_, {r___, {w___, x_, v___}, s___}} /; FreeQ[{w, v}, x | -x] :→
      {gm = DeleteCases[g, x], rm = Sort[DeleteCases[Map[ReduceWord, {r, s} /.
            {-x :→ Apply[Sequence, {v, w}], x :→ Apply[Sequence, -Reverse[{v, w}]]}], {}]]}}];
   {gm, rm = Union[Map[NormWord, rm]]}, {Length[gm], Length[rm]}]]

WordForm[w_List] :=
 Apply[SequenceForm, ReplaceRepeated[ReduceWord[w] /. {-x_ :→ x^(-1)},
    {a___, x_^(e_ : 1), x_^(f_ : 1), b___} :→ {a, x^(e + f), b}]] /. x_^n_ :→ Superscript[x, n]]

PresentationForm[{gen_, rel_}] := {gen, Map[WordForm, rel]}
```

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